A METHOD OF DESIGNING
CONSTANT-PHASE NETWORKS

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ABSTRACT

The purpose of this study was to formulate design information for the determination of a minimum-phase network whose transfer function has a nearly-constant phase angle throughout a given frequency range.

The method used is the cascading of similar RC network sections so that their phase characteristics add to produce a total characteristic which falls within the tolerance band. The basic network section chosen has a bell-shaped phase characteristic which contributes to the over-all phase throughout a portion of the required frequency band and approaches zero elsewhere.

By centering an infinite number of identical characteristics at equal intervals on a logarithmic frequency scale, a periodic rippling phase function is obtained. The corresponding network contains an infinite number of sections and has an unlimited bandwidth. The question of bandwidth is thus temporarily deferred. Graphical and analytical relationships relating the pole-zero positions of the network sections, phase angle, and phase tolerance are presented. The accuracy of the results is discussed.

The phase characteristic for a practical network is derived from the infinite-band function by removing all sections whose phase characteristic centers lie outside the

desired frequency band. Sections whose centers lie well outside the frequency band have a negligible effect inside the band. However, removal of sections which are centered near the band edges causes the phase characteristic to sag in that vicinity. This undesirable effect is studied, and an analytical method of correcting it is developed.

The dependency of bandwidth on the number of sections retained and on the correction procedure is investigated. Results are presented in the form of a graph which relates phase tolerance, bandwidth, and the number of network sections.

It is pointed out that a resulting pole-zero plot, which will in every case consist of poles and zeros alternating on the negative real frequency axis, may be interpreted as any appropriate transfer or driving point function. A network which yields that function may be developed by modern synthesis techniques. Modern synthesis techniques for realizing alternate two-terminal-pair networks from the pole-zero plot are discussed briefly.

Upon completion of the design information, the design procedure is summarized; and one network example is presented.

The design information covers a range of phase angles from $-90°$ to $+90°$. Detailed information for tolerances as low as $\pm 0.015°$ for a phase angle of $45°$ is given. Smaller tolerance designs are possible for angles other than 45°.

The accuracy of this design method depends on certain approximations made in the development of relations to describe the infinite-band phase function and on the sag correction procedure. All inaccuracies caused by approximations are such as to result in a smaller phase tolerance. For tolerances greater than ±0.1°, the maximum possible excursion (caused by the correction procedure) from the tolerance band is one per cent of the tolerance. Larger deviations may occur near the edges of the frequency band for tolerances less than ±0.1°.

CHAPTER I

INTRODUCTION

The Problem.--Two-terminal-pair networks which provide a linear attenuation with constant phase shift over a wide frequency range are often required in communication and control systems. They are particularly important in shaping the phase character of the forward gain in feedback systems. Herein is presented the design information necessary for realizing such a network. The design parameters around which the design information is developed are: (a) a specified phase angle, (b) a specified tolerance, and (c) a specified frequency range.

Historical Background.--A constant-phase network may be classified as a type of equalization network. In general, the purpose of an equalization network is to shape or modify a given attenuation or phase characteristic. Many well-known general methods for designing equalization networks have been published. An excellent outline of these is given by Saraga (1). Most of these methods, although applicable to the design of constant-phase networks, are crude from the phase standpoint because they are formulated with attenuation equalization as the primary concern. Certain graphical methods may be directly applied, but are objectionable because they utilize trial and error. A method presented by Saraga (1) eliminates trial and error but, for the problem under consideration, would require the use of mechanical, optical, or electrical equipment.

Certain methods of designing phase splitting networks appear to be applicable with proper modification. Orchard (2), Darlington (3), and Saraga (4) use elliptic functions to obtain Chebycheff approximations to a constant phase difference between the output terminals of two networks which have a common input. These methods, if they can be modified for the design of a single two-port network, would provide exact solutions to the problem. Further investigation in this respect appears promising.

The method of approach proposed here is similar to that used by Morrison (5) in designing a constant-argument RC driving-point admittance. He uses a canonic RC network containing an infinite number of branches to satisfy tolerance and average phase requirements. A practical circuit is determined by dropping all branches except those which provide constant phase within a given bandwidth. He was able to develop approximate mathematical expressions in closed form for phase angle and tolerance, which makes the method practical. Advantages of the method are that very little trial and error procedure is required, and the ease of application is not dependent on the complexity of the network. The principal difference between Morrison's procedure and that used in this study is that his development is based on a summation of an infinite number of basic admittance functions, whereas the method presented here is based on the product of an infinite number of basic transfer functions.

Method and Scope.--Basically two approaches are possible for the solution of this problem. The first is to derive a phase function which fits the tolerance band. This function is then used to synthesize a network. The second, which is used in this study, is to start with a network and adjust it until its characteristic fits the tolerance band. In particular, the required function is built up by cascading network sections so that the attenuation and phase characteristics of the different sections are additive. By this means the problem of network realization is circumvented. Of course, the use of modern network synthesis techniques to realize the transfer function derived in this manner is not ruled out.

A semi-mathematical and systematic approach is made possible by first considering an infinite number of similar network sections connected in cascade with isolating amplifiers. Phase and tolerance requirements are thus satisfied over an unlimited frequency range. The desired bandwidth may then determine the number of sections to retain for a practical network.

The attenuation characteristic is considered of secondary importance in this problem and is therefore not discussed. The attenuation can be easily determined in any particular case from a knowledge of the pole-zero plot by simple methods given in any good control handbook (6).

CHAPTER II

THE BASIC NETWORK SECTION

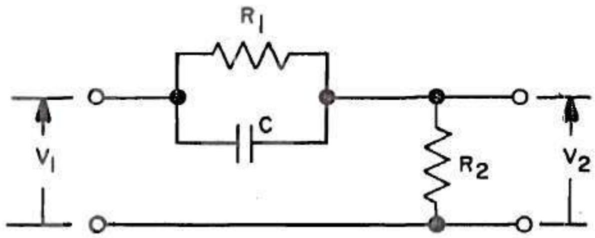
It is desired to obtain a phase function that is nearly constant over an infinite frequency range by summing phase curves which are shifted replicas of some standard phase curve. For this to be possible, the standard curve must contribute constructively toward the desired curve over some frequency range and must have little effect outside this frequency range.

Simple networks possessing phase characteristics which have these properties are illustrated in Figs. 1 and 2. Except for a constant multiplier, the function in Fig. 1(b) is the reciprocal of the function in Fig. 2(b). The phase of one network is merely the negative of the phase of the other. Thus, a function derived by using one of the two network configurations as a basis could be used to construct either of the two types of networks. In accordance with this realization, all further development will be concerned only with the network of Fig. 1.

A few useful observations concerning the basic phase characteristic should be made. First, note that according to Fig. 1(b), $k \geq 1$.

Therefore $$k^{1/2} - k^{-1/2} \geq 1$$

(a)



(b)

$$\frac{V_2}{V_1} = \frac{s + \sigma_i k^{-1/2}}{s + \sigma_i k^{1/2}}$$

$$\text{WHERE} \quad k = \frac{R_1 + R_2}{R_2}$$

$$\sigma_i = \frac{k^{1/2}}{R_1 C}$$

(c)

$$\beta(\omega) = \arg\left(\frac{V_2}{V_1}\right) = \tan^{-1}\left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega}{\sigma_i} + \frac{\sigma_i}{\omega}}\right)$$

(d)

(e)

(f)

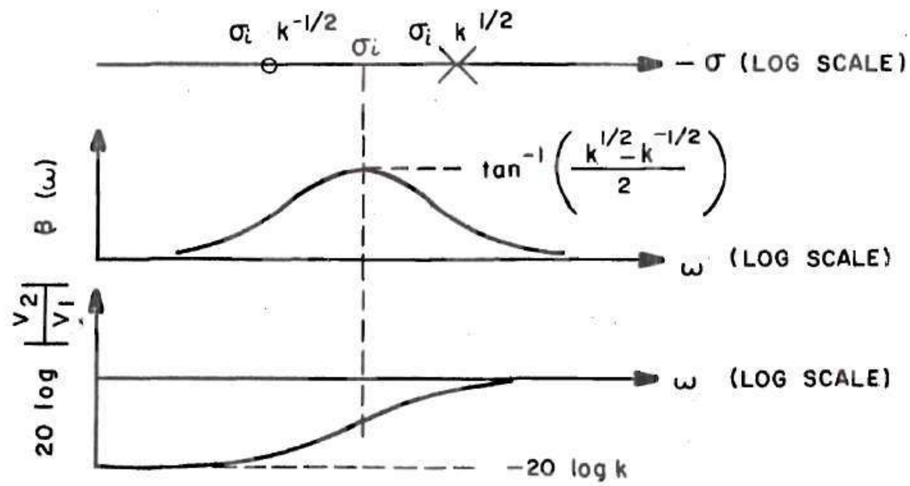
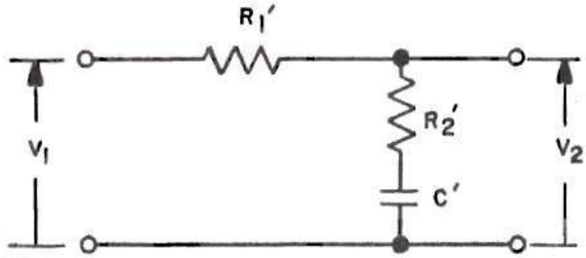


Fig. 1. Description of Basic Network for Positive Phase.

(a)



(b)

$$\frac{V_2}{V_1} = \frac{1}{k}\left(\frac{s + \sigma_i\, k^{1/2}}{s + \sigma_i\, k^{-1/2}}\right)$$

$$\text{WHERE}\quad k = \frac{R_1' + R_2'}{R_2'}$$

$$\sigma_i = \frac{k^{1/2}}{(R_1' + R_2')\,C'}$$

(c)

$$\beta(\omega) = \arg\left(\frac{V_2}{V_1}\right) = -\tan^{-1}\left(\frac{k^{1/2} - k^{-1/2}}{\dfrac{\omega}{\sigma_i} + \dfrac{\sigma_i}{\omega}}\right)$$

(d)

(e)

(f)

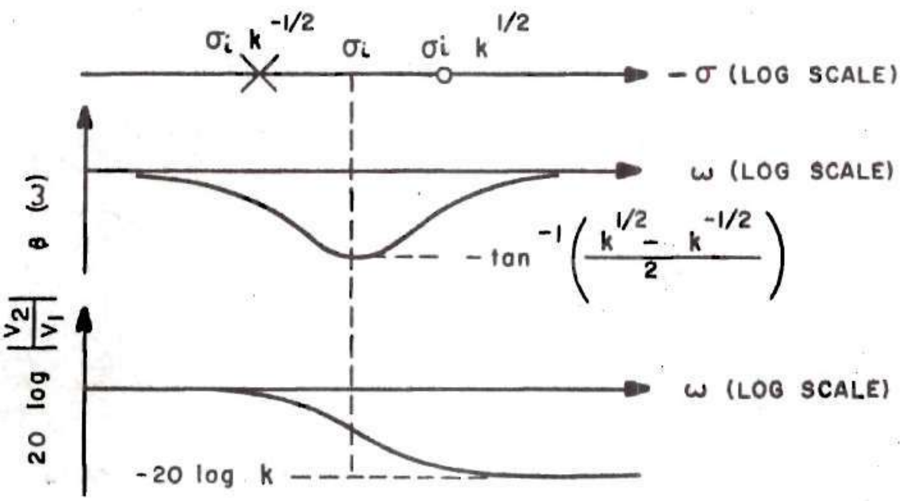


Fig. 2. Description of Basic Network for Negative Phase.

and $$0 < \beta(\omega) < 90°$$

Secondly, on a logarithmic frequency scale, the curve is symmetrical about its maximum point. This may be simply demonstrated through the following change in variable:

Let $$u = \log \frac{\omega}{\sigma_i}$$

Then $$\frac{\omega}{\sigma_i} = 10^u$$

Fig. 1(c) becomes

$$\beta(u) = \tan^{-1}\left(\frac{k^{1/2} - k^{-1/2}}{10^u - 10^{-u}}\right) \qquad (1)$$

It is obvious from this equation that

$$\beta(u) = \beta(-u)$$

It remains to show that $\beta(u)$ is maximum at $u = 0$. This is accomplished by noting that $\beta(u)$ is maximum when $10^u + 10^{-u}$ is minimum and that

$$(10^u - 1)^2 \geq 0$$

or $$10^{2u} + 1 \geq 2(10^u)$$

Thus $$10^u + 10^{-u} \geq 2$$

and $$(10^u + 10^{-u})_{u=0} = 2$$

Finally, the factor k completely characterizes the shape of the phase curve on a logarithmic frequency scale. This fact is made clear by noting that equation 1, which is a function of the logarithmic frequency variable u, does not contain the factor σ_1. In other words, the magnitude of the phase depends only on the pole position relative to the zero position. The factor σ_1 effects only the frequency at which the curve is centered. By changing σ_1 while keeping k constant, the curve may be shifted along the logarithmic frequency axis without changing its shape.

Ultimately, element values will be needed in terms of σ_1 and k. In order to accomplish this, one of the element values must be fixed by some additional requirement. One alternative is to normalize the elements in Fig. 1 with respect to $R_1 + R_2$ and the elements in Fig. 2 with respect to $R_1' + R_2'$. In mathematical form,

let

$$R_1 + R_2 = 1$$

and

$$R_1' + R_2' = 1$$

The element values are found to be as follows:

$$R_1 = \frac{k-1}{k} \quad , \quad R_2 = \frac{1}{k} \quad , \quad C = \frac{k^{3/2}}{(k-1)\sigma_1} \qquad (2)$$

$$R_1' = \frac{k-1}{k} \quad , \quad R_2' = \frac{1}{k} \quad , \quad C = \frac{k^{1/2}}{\sigma_1} \qquad (3)$$

In the sections that follow, design information will be developed in terms of pole and zero positions, with the pole-zero plot in Fig. 1(d) as the basic configuration.

CHAPTER III

THE INFINITE-BAND PHASE FUNCTION

The magnitude and phase functions for an infinite number of equally spaced pole-zero pairs may be written

$$\left|\frac{V_2}{V_1}\right| = \prod_{i=-\infty}^{\infty} \left|\frac{s + n^i k^{-1/2}}{s + n^i k^{1/2}}\right| \qquad (4)$$

$$\phi(k,n,\omega) = \sum_{i=-\infty}^{\infty} \tan^{-1}\left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega}{n^i} + \frac{n^i}{\omega}}\right) \qquad (5)$$

where $n^i = \sigma_i = \sigma_1^i$

and $i = o, \pm 1, \pm 2, \ldots$

These equations are illustrated graphically in Fig. 3. The phase function ϕ is a periodic function of $\log \omega$ (see Appendix I), and its variation can be described in terms of an "average" value

$$\phi_m = \frac{\phi_{max} + \phi_{min}}{2}$$

and a tolerance $\frac{\Delta\phi}{2} = \frac{\phi_{max} - \phi_{min}}{2}$

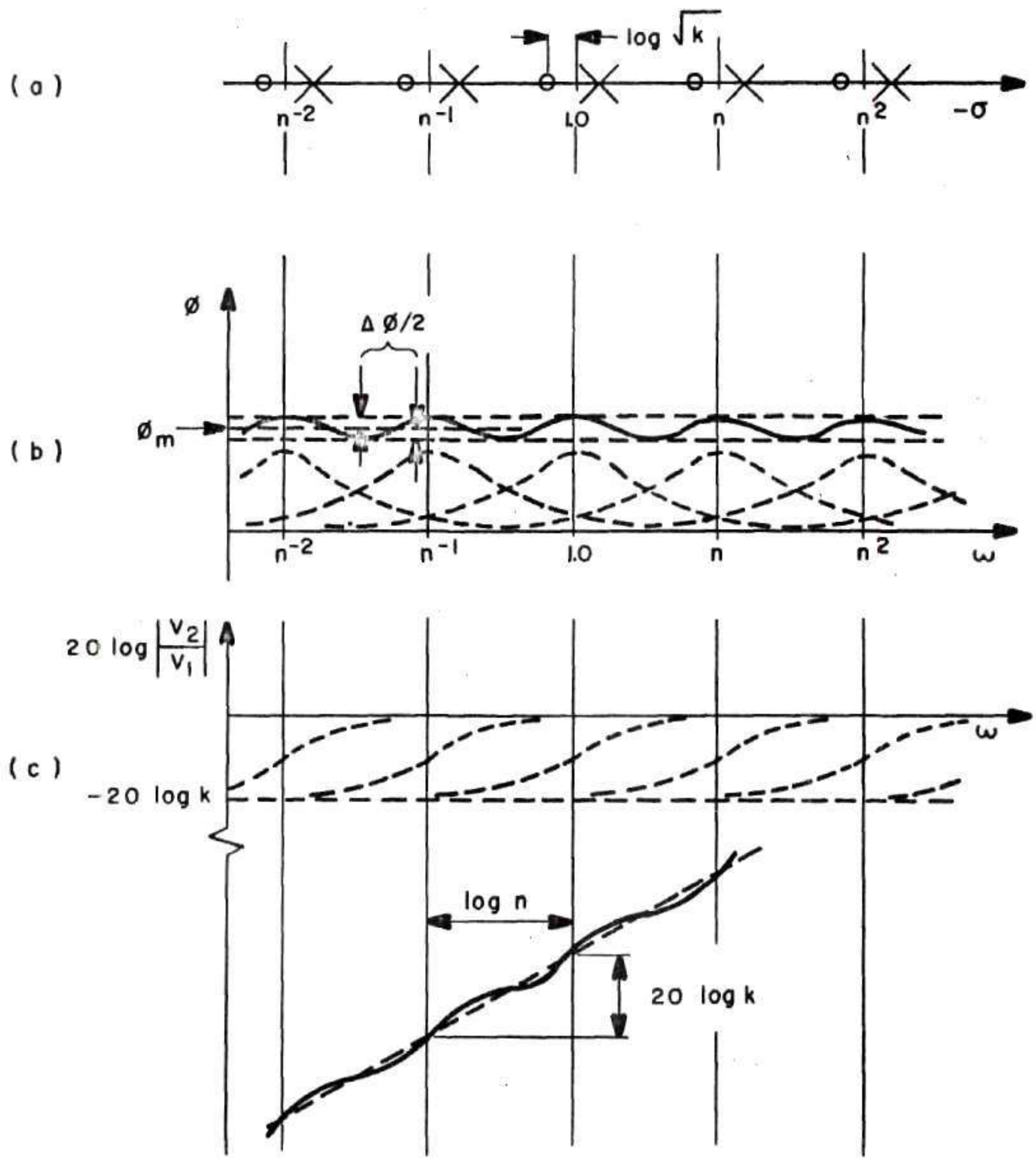


Fig. 3. The Infinite-Band Transfer Function.
(a) Pole-Zero Plot. (b) Phase Characteristic. (c) Attenuation Characteristic.

It is desired to develop a method for obtaining the values of n and k such that specified values of ϕ_m and $\Delta\phi$ are realized.

<u>Basic Relationships</u>.--Bode (7) has shown that for a minimum phase function with linear attenuation, the phase shift will be constant and proportional to the slope of the attenuation characteristic. The mathematical expression is

$$\theta = \frac{\pi}{12.04} \frac{d\gamma}{du} \tag{6}$$

where θ is the phase shift measured in radians, γ is the attenuation measured in decibels, and $\frac{d\gamma}{du}$ is the attenuation slope measured in decibels per octave. The average slope of the attenuation characteristic may be found by noting that each pole-zero pair adds a maximum of -20 log k decibels to the attenuation, and each pole-zero pair is separated by log n decades. This is indicated in Fig. 3(c). With conversion from decades to octaves, the slope is given by

$$\frac{d\gamma}{du} = \left(\frac{20 \log k}{\log n}\right)\left(\log 2\right)$$

Make this substitution for $d\gamma/du$ in equation 6 to obtain

$$\theta(\text{Radians}) = 1.57 \frac{\log k}{\log n} \tag{7}$$

or

$$\theta(\text{Degrees}) = 90° \frac{\log k}{\log n} \tag{8}$$

From equation 8 the expression for k is

$$k = n^{\theta/90} \tag{9}$$

The validity of equation 9 may be checked easily for the limits of θ. If $\theta = 0°$, then $k = 1$; and the phase function (equation 5) becomes

$$\phi(1,n,\omega) = 0°$$

Thus, $$\phi_m = \phi = \theta = 0°$$

If $\theta = 90°$, then $k = n$; and the phase function can be evaluated as follows. Equation 5 may be written as

$$\phi(k,n,\omega) = \lim_{M\to\infty} \sum_{i=-M}^{M} \left[\tan^{-1} \frac{\omega k^{1/2}}{n^i} - \tan^{-1} \frac{\omega}{n^i k^{1/2}} \right]$$

Make the substitution $k = n$. Then,

$$\phi(n,n,\omega) = \lim_{M\to\infty} \sum_{i=-M}^{M} \left[\tan^{-1} \frac{\omega}{n^{i-1/2}} - \tan^{-1} \frac{\omega}{n^{i+1/2}} \right]$$

Expand this equation and cancel like terms to obtain

$$\phi(n,n,\omega) = \lim_{M\to\infty} \left[\tan^{-1} \frac{\omega}{n^{-M-1/2}} - \tan^{-1} \frac{\omega}{n^{M+1/2}} \right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0)$$

$$= 90°$$

Thus, $\phi_m = \phi = \theta = 90°$

As demonstrated above, ϕ_m and θ are equal for a phase angle of 0° or 90°. Ideally, θ and ϕ_m would always be equal. It will be shown later that they are equal for only one other angle, 45°. With the anticipation that θ and ϕ_m will at least be approximately equal, θ will be used as a variable in place of k in equation 5.

Eventually, data must be obtained for values of θ from 0° to 90°. It may now be shown that data obtained for the range 0° to 45°, if properly interpreted, will be the same as corresponding data for the range 45° to 90°. Toward this end consider the two pole-zero plots shown in Fig. 4. Equation 9 applies to Fig. 4(a), whereas for Fig. 4(b),

$$k' = n^{\frac{90-\theta}{90}} = \frac{n}{k} \qquad (10)$$

In Fig. 4(a) the pole-zero pairs are centered at n^i, and in (b) they are centered at $n^{i\pm 1/2}$. The equations for the phase angles of (a) and (b) are respectively,

$$\phi_a = \sum_{i=-\infty}^{\infty} \left[\tan^{-1} \left(\frac{\omega k^{1/2}}{n^i}\right) - \tan^{-1} \left(\frac{\omega}{n^i k^{1/2}}\right) \right] \qquad (11)$$

and $$\phi_b = \sum_{i=-\infty}^{\infty} \left[\tan^{-1} \left(\frac{\omega k'^{1/2}}{n^{i+1/2}}\right) - \tan^{-1} \left(\frac{\omega}{n^{i+1/2} k'^{1/2}}\right) \right] \qquad (12)$$

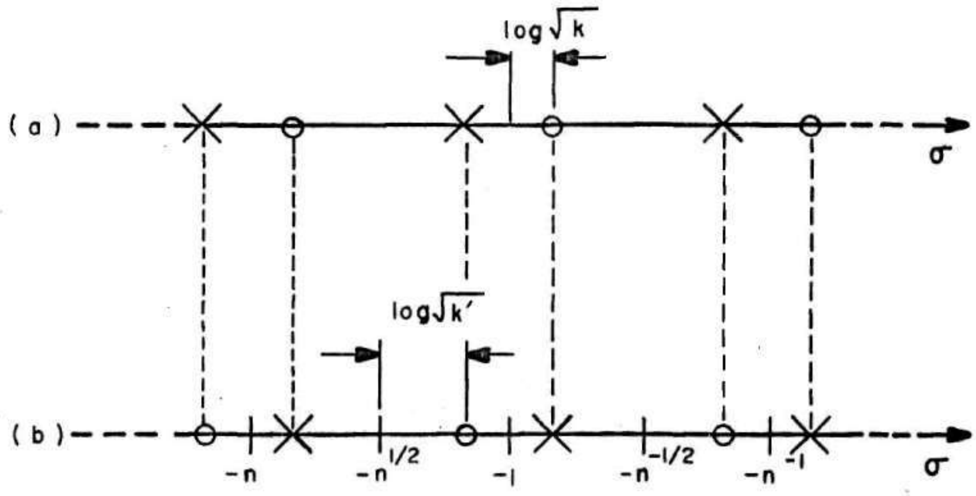


Fig. 4. (a) Pole-Zero Configuration for θ.
(b) Pole-Zero Configuration for $90° - \theta$.

Substitute for k' from equation 10. Equation 12 becomes

$$\phi_b = \sum_{i=-\infty}^{\infty} \left[\tan^{-1}\left(\frac{\omega}{k^{1/2}n^i}\right) - \tan^{-1}\left(\frac{\omega k^{1/2}}{n^{i+1}}\right) \right] \tag{13}$$

The addition of equations 11 and 13 yields

$$\phi_a+\phi_b = \sum_{i=-\infty}^{\infty} \left[\tan^{-1}\left(\frac{\omega k^{1/2}}{n^i}\right) - \tan^{-1}\left(\frac{\omega k^{1/2}}{n^{i+1}}\right) \right]$$

This may be written as

$$\phi_a+\phi_b = \lim_{M\to\infty} \sum_{i=-M}^{i=M} \left[\tan^{-1}\left(\frac{\omega k^{1/2}}{n^i}\right) - \tan^{-1} \frac{\omega k^{1/2}}{n^{i+1}} \right]$$

Expand and cancel like terms to obtain

$$\phi_a+\phi_b = \lim_{M\to\infty} \left(\tan^{-1} \frac{\omega k^{1/2}}{n^{-M}} - \tan^{-1} \frac{\omega k^{1/2}}{n^{M+1}} \right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0)$$

$$= 90°$$

Restated in a more descriptive form,

$$\phi(\theta,n,\omega) = 90° - \phi(90-\theta,n,\omega) \tag{14}$$

It is important to remember that this equation applies only when the poles (zeros) for $\phi(\theta,n,\omega)$ coincide with the zeros (poles) for $\phi(90-\theta,n,\omega)$.

It follows from equation 14 that

$$\Delta\phi(\theta,n) = \Delta\phi(90-\theta,n) \tag{15}$$

and
$$\phi_m(\theta,n)-\theta = (90-\theta) - \phi_m(90-\theta,n) \tag{16}$$

Since equations 15 and 16 are independent of ω, they are valid even if one pole-zero configuration is shifted with respect to the other.

Phase Tolerance.--In order to gain some insight as to the behavior of the phase curve and to provide data with which to check subsequent developments, equation 17 was evaluated

$$\phi(\theta,n,\omega) = \sum_{i=-\infty}^{\infty} \tan^{-1}\left[\frac{n^{\theta/180} - n^{-\theta/180}}{\frac{\omega}{n^i} + \frac{n^i}{\omega}}\right] \tag{17}$$

for several values of θ,n,ω. This information, accompanied by $\Delta\phi$ and $\phi_m-\theta$, is given in Table 1[1] of Appendix II. The values of ω, 1 and $n^{1/2}$, were chosen to give ϕ_{max} and ϕ_{min}, which occur at n^i and $n^{i\pm1/2}$ respectively with pole-zero pairs centered at $\omega = n^i$. Data are not included for values of θ from 45° to 90°, since equations 14, 15, and 16 may be used in connection with Table 1 to obtain this.

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[1]This and all other data tabulated in Appendix II were obtained through the use of the Bell General Purpose System on an IBM 650 Computer.

A concise way of representing the interrelationship of $\Delta\phi$, θ, and n is suggested by the following development. $\Delta\phi$ is defined as

$$\Delta\phi = \phi_{max}(\theta,n) - \phi_{min}(\theta,n)$$

According to equation 14, ϕ_{min} may be written

$$\phi_{min}(\theta,n) = 90° - \phi_{max}(90-\theta,n)$$

With this substitution for $\phi_{min}(\theta,n)$, $\Delta\phi$ becomes

$$\Delta\phi = \phi_{max}(\theta,n) + \phi_{max}(90-\theta,n) - 90°$$

or

$$\Delta\phi = \sum_{i=-\infty}^{\infty}\left[\tan^{-1}\frac{k^{1/2}}{n^i} - \tan^{-1}\frac{1}{n^i k^{1/2}} + \tan^{-1}\frac{k'^{1/2}}{n^i} - \tan^{-1}\frac{1}{n^i k'^{1/2}}\right] - 90°$$

Make the substitution for k' indicated by equation 10, and rearrange terms to obtain

$$\Delta\phi = \sum_{i=-\infty}^{\infty}\left[\tan^{-1}\frac{k^{1/2}}{n^i} - \tan^{-1}\frac{k^{1/2}}{n^{i+1/2}}\right]$$

$$+ \sum_{i=-\infty}^{\infty}\left[\tan^{-1}\frac{1}{k^{1/2}n^{i-1/2}} - \tan^{-1}\frac{1}{k^{1/2}n^i}\right] - 90°$$

Combine terms within each set of brackets and transform i to -i in the last summation. The result is

$$\Delta\phi = \sum_{i=-\infty}^{i=\infty} \tan^{-1}\left[\frac{n^{1/4} - n^{-1/4}}{\frac{n^{i+1/4}}{k^{1/2}} + \frac{k^{1/2}}{n^{i+1/4}}}\right]$$

$$+ \sum_{i=-\infty}^{\infty} \tan^{-1}\left[\frac{n^{1/4} - n^{-1/4}}{\frac{n^{i+1/4}}{k^{1/2}} + \frac{k^{1/2}}{n^{i+1/4}}}\right] - 90°$$

or $$\Delta\phi = \sum_{i=-\infty}^{i=\infty} \tan^{-1}\left[\frac{n^{1/4} - n^{-1/4}}{\frac{k^{1/2}n^{-1/4}}{n^{i}} + \frac{n^{i}}{k^{1/2}n^{-1/4}}}\right] - 90° \qquad (18)$$

Now, compare the form of the summation term in equation 18 with the form of equation 5. It is apparent that the summation term may be interpreted as the value of the infinite-band phase function for $k = n^{1/2}$ and $\omega = k^{1/2}n^{-1/4}$. The pole-zero pairs are centered at n^{i} as before. Accordingly, equation 18 may be written as follows:

$$\frac{\Delta\phi(\theta,n)}{2} = \phi(\theta_o,n,\omega_o) - 45° \qquad (19)$$

where $$\omega_o = \frac{k^{1/2}}{n^{1/2}} = n^{\frac{\theta-45}{180}}$$

and $$\theta_o = 90\,\frac{\log n^{1/2}}{\log n} = 45°$$

If $\theta = 45°$, then $\omega_o = 1$, which corresponds to a maximum point of $\phi(\theta_o, n, \omega_o)$. For this case equation 19 becomes

$$\frac{\Delta\phi(45°,n)}{2} = \phi_{max}(45°,n) - 45°$$

rearranging, $$\phi_{max}(45°,n) - \frac{\Delta\phi(45°,n)}{2} = 45°$$

The left hand side of this equation is the angle $\phi_m(45°,n)$.

Thus, $$\phi_m(45°,n) = 45°$$

It is evident that equation 19 gives the deviation of $\phi(45°,n,\omega_o)$ from its "average" angle $\phi_m(45°,n)$.

The following question now arises: Can $\phi(45°,n,\omega_o)-45°$ be normalized so that a single curve may be used to represent the ripple about the angle 45°? One such normalization is accomplished by plotting $\Delta\phi(\theta,n)/\Delta\phi(45°,n)$ or

$$\frac{2[\phi(45°,n,\omega_o) - 45°]}{\Delta\phi(45°,n)} \tag{20}$$

versus $\log \omega_o$, where ω_o extends from $n^{-1/4}$ to n^o. Or an equivalent procedure is to plot quantity 20 versus θ, where θ extends from 0° to 45°. The results of a computer program written for this purpose are illustrated in Fig. 5. All lines intersect at $\theta = 0°$ and at $\theta = 45°$. Elsewhere they depart from coincidence only slightly, the maximum difference being 0.0065° near $\theta = 22.5°$. The data in Table 2

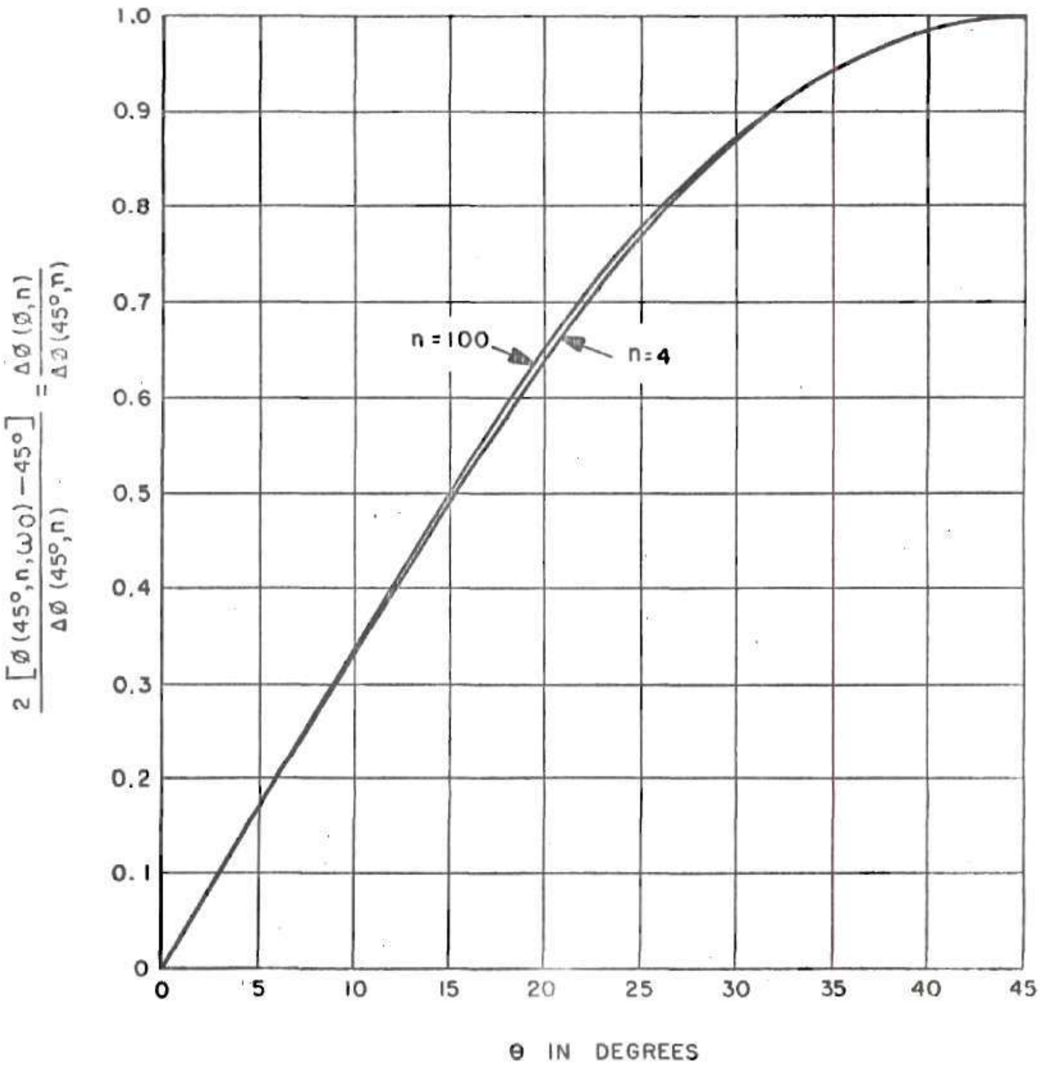


Fig. 5. Ratio of $\Delta\emptyset(\theta,n)$ to $\Delta\emptyset(45°,n)$ as a Function of θ.

(Appendix II), from which Fig. 5 was plotted, show that for $n \leq 20$ the results agree to three decimal places for a given value of θ.

A further simplification can be made by noting that the curves in Fig. 5 resemble part of a sine wave. Accordingly, one can approximate equation 19 with

$$\Delta\phi(\theta,n) = \Delta\phi(45°,n)\sin 2\theta \tag{21}$$

Fig. 6 illustrates the error in $\Delta\phi(\theta,n)$ resulting from this approximation. The value of $\Delta\phi(\theta,n)$ predicted by equation 21 will be smaller than the actual value by the amount shown in Fig. 6. The corresponding data in Table 3 (Appendix II) show that the error is less than 0.0003° for $n \leq 10$.

As indicated by the positive values of $\phi_m - \theta$ in Table 1, $\phi_m \geq \theta$. Thus,

$$\sin 2\phi_m \geq \sin 2\theta$$

for $$0 < \theta < 45°$$

This means that the use of ϕ_m instead of θ in equation 21 will algebraically decrease the difference, $\Delta\phi$ (actual) - $\Delta\phi$ (calculated).

The suggested equation is

$$\Delta\phi(\theta,n) = \Delta\phi(45°,n)\sin 2\phi_m \tag{22}$$

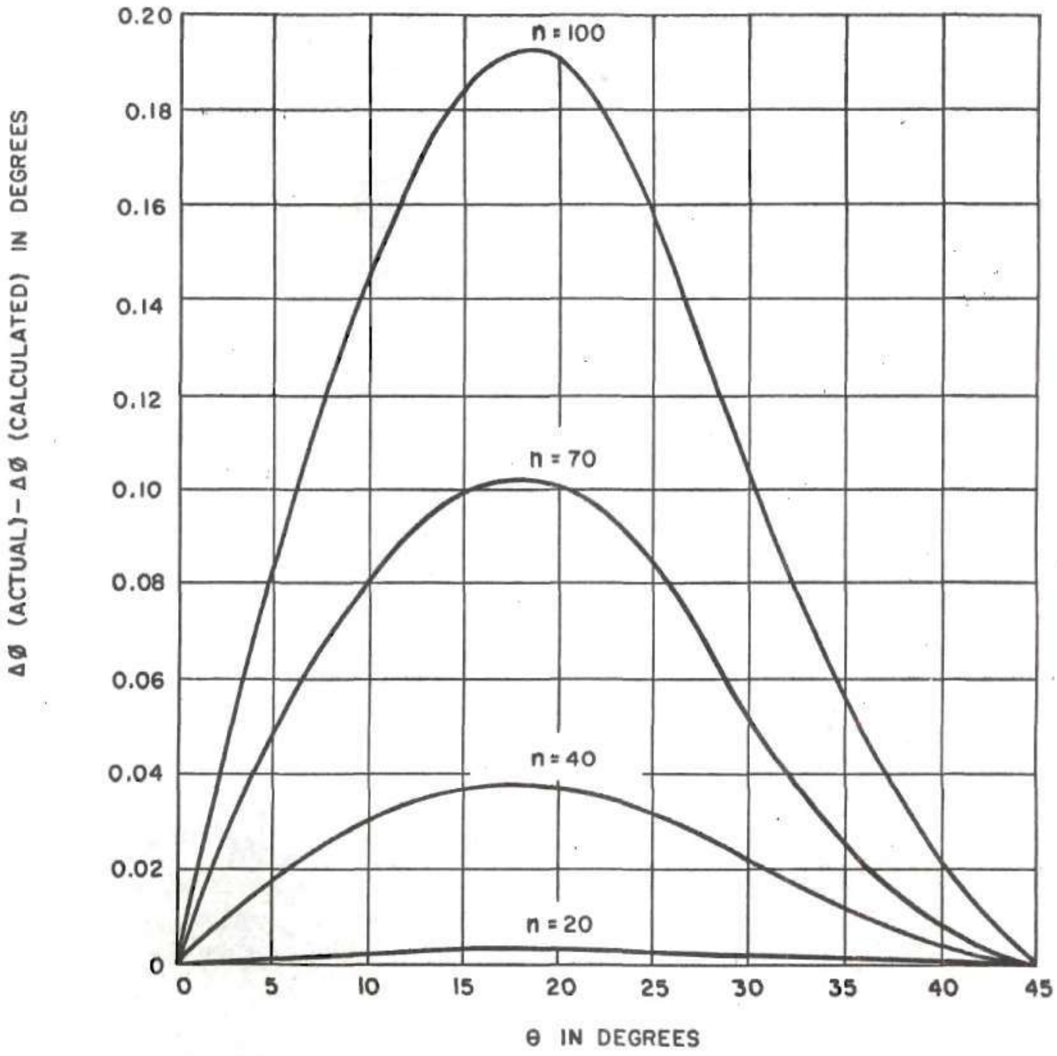


Fig. 6. Error in $\Delta\phi$ Caused by the Approximation of $\Delta\phi(\theta,n)/\Delta\phi(45°,n)$ with Sin 2θ.

Fig. 7 illustrates the error in $\Delta\phi(\theta,n)$ resulting from the use of equation 22. This error, although almost twice as great as that shown in Fig. 6, is more easily tolerated because it is opposite in sign to that in Fig. 6. If equation 22 is used, the resulting value of $\Delta\phi$ will be less than or equal to that desired. Equation 22 also has the advantage that ϕ_m will be known in a practical situation, whereas θ will have to be determined.

The remaining requirement for use of equations 21 and 22 is a table or graph relating $\Delta\phi(45°,n)$ and n. The graph is given in Fig. 8. Table 5, in Appendix II permits interpolation accurate to 0.002° or better for $\Delta\phi(45°,n) \leq 8°$.

In conclusion, the best procedure for determining n, given ϕ_m and $\Delta\phi$, is:

1. Calculate $\Delta\phi(45°,n)$ using equation 22.
2. Use the calculated value of $\Delta\phi(45°n)$ to find n from Fig. 8 or Table 5.

Correlation between θ and ϕ_m.--From the previous discussion and from Table 1 in Appendix I, it is apparent that θ and ϕ_m are equal for the angles 0°, 45°, and 90° exclusively. The data in Table 1 also indicate that

$$\phi_m(\theta,n) - \theta \approx \phi_m(45-\theta,n) - (45-\theta)$$

Furthermore, $\phi_m(\theta,n) - \theta$ increases monotonically as θ varies from 0° to 22.5°. This suggests that an

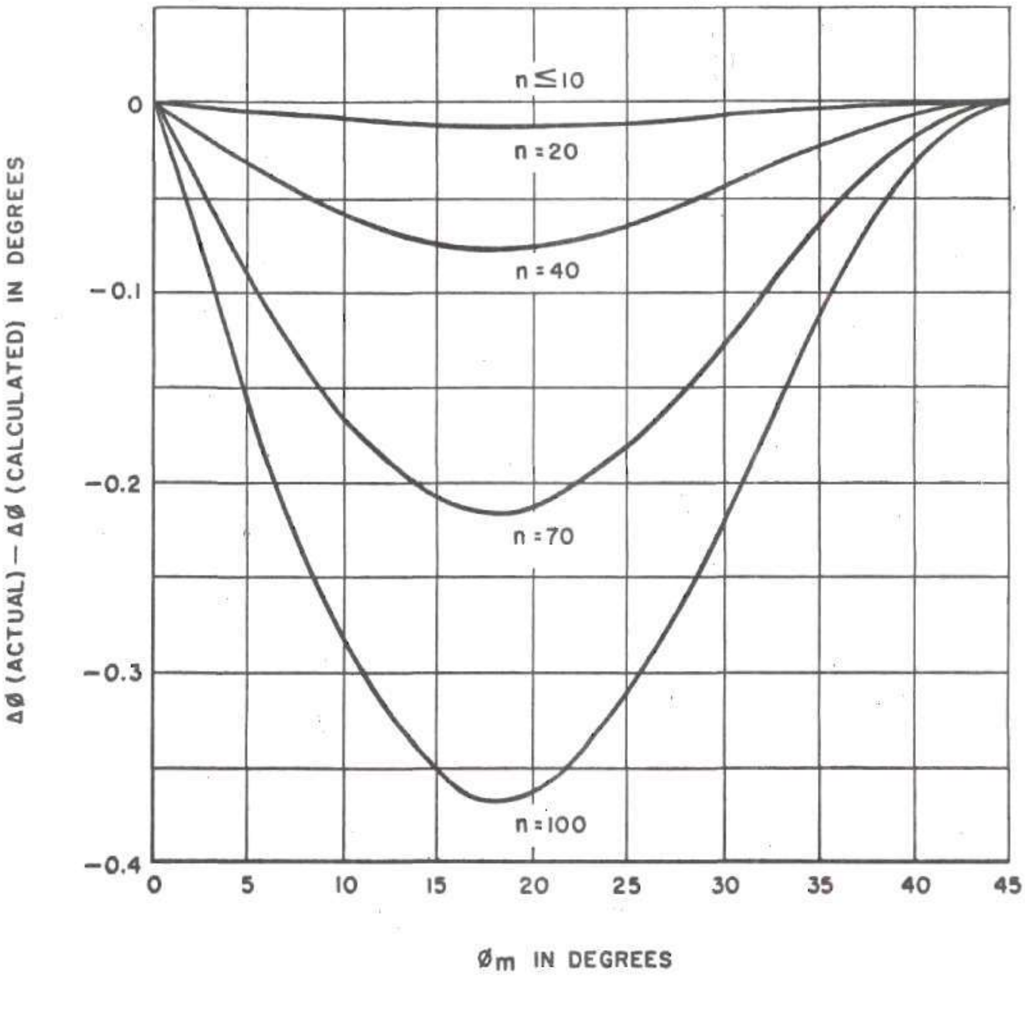


Fig. 7. Error in $\Delta\phi$ Caused by the Approximation of $\Delta\phi(\theta,n)/\Delta\phi(45°,n)$ with Sin $2\phi_m$.

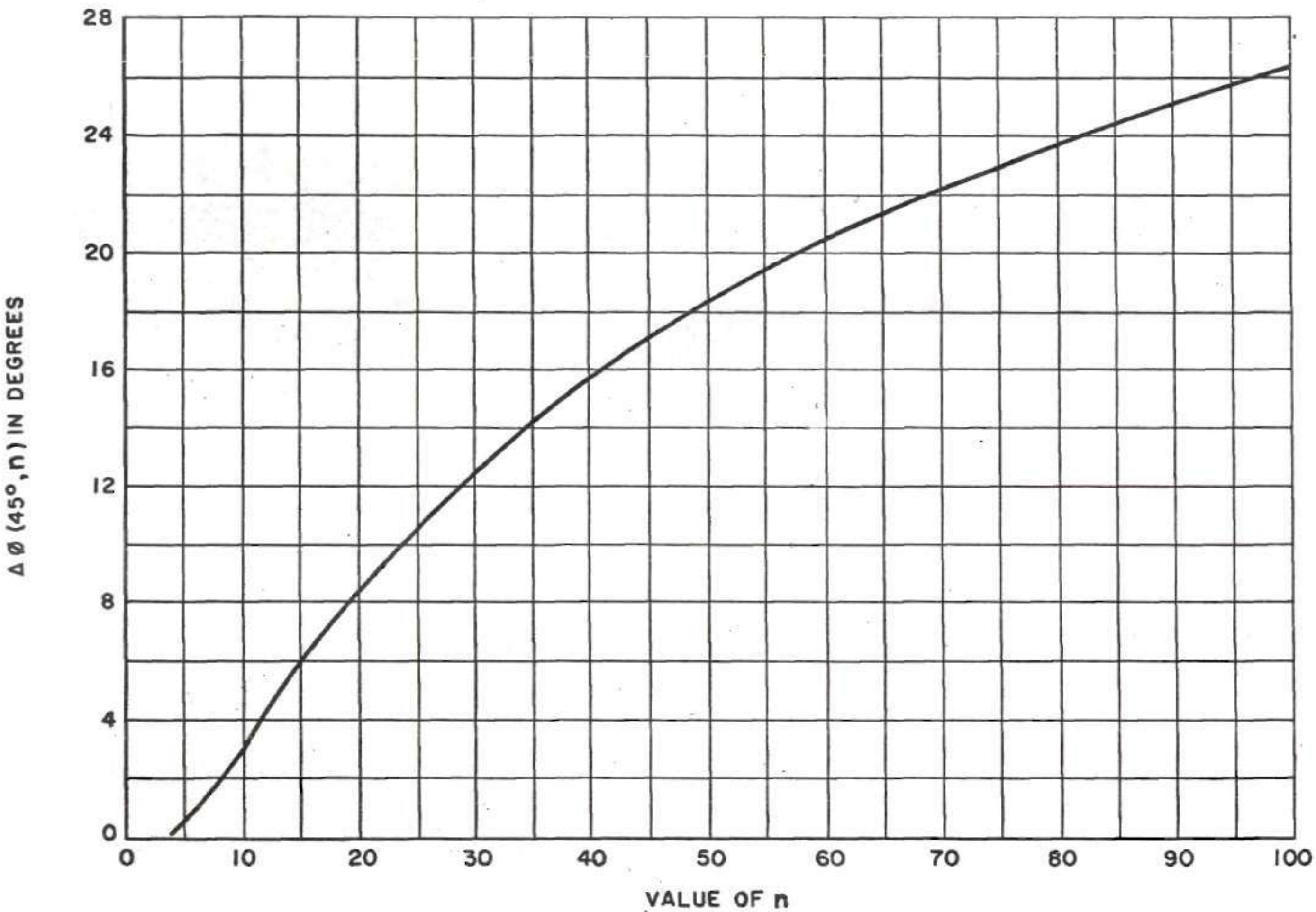


Fig. 8. Variation of $\Delta\phi$ with n, for $\theta = 45°$.

approximation similar to that found in the previous section might be possible. The approximation to be tried here is

$$\phi_m(\theta,n)-\theta = \left[\phi_m(22.5°,n)-22.5°\right] \sin 4\theta \qquad (23)$$

In Fig. 9 $\phi_m(22.5°,n)-22.5°$ is given as a function of n. Table 6 (Appendix II), from which Fig. 9 was drawn, includes enough data to allow values of $\phi_m - \theta$ accurate to within $\pm$ 0.0002° for a given value of n.

Fig. 10 illustrates the accuracy of equation 23. Note that the difference between calculated and actual values of ϕ_m is the same as the difference between calculated and actual values of $\phi_m - \theta$. From Fig. 10 it is clear that the approximation is correct to four decimal places for $n \leq 20$.

In a practical situation, n will have been determined from Fig. 8 and equation 22. From this value of n and a given value of ϕ_m, θ may be determined through the use of equation 23 and Fig. 9 by successive approximations. For the first approximation, $\sin 4\phi_m$ may be used in place of $\sin 4\theta$ in equation 23.

The values of θ and n may be substituted into equation 9 to find k. The resulting phase function will have a value of ϕ_m differing from the expected value by the amount indicated in Fig. 10.

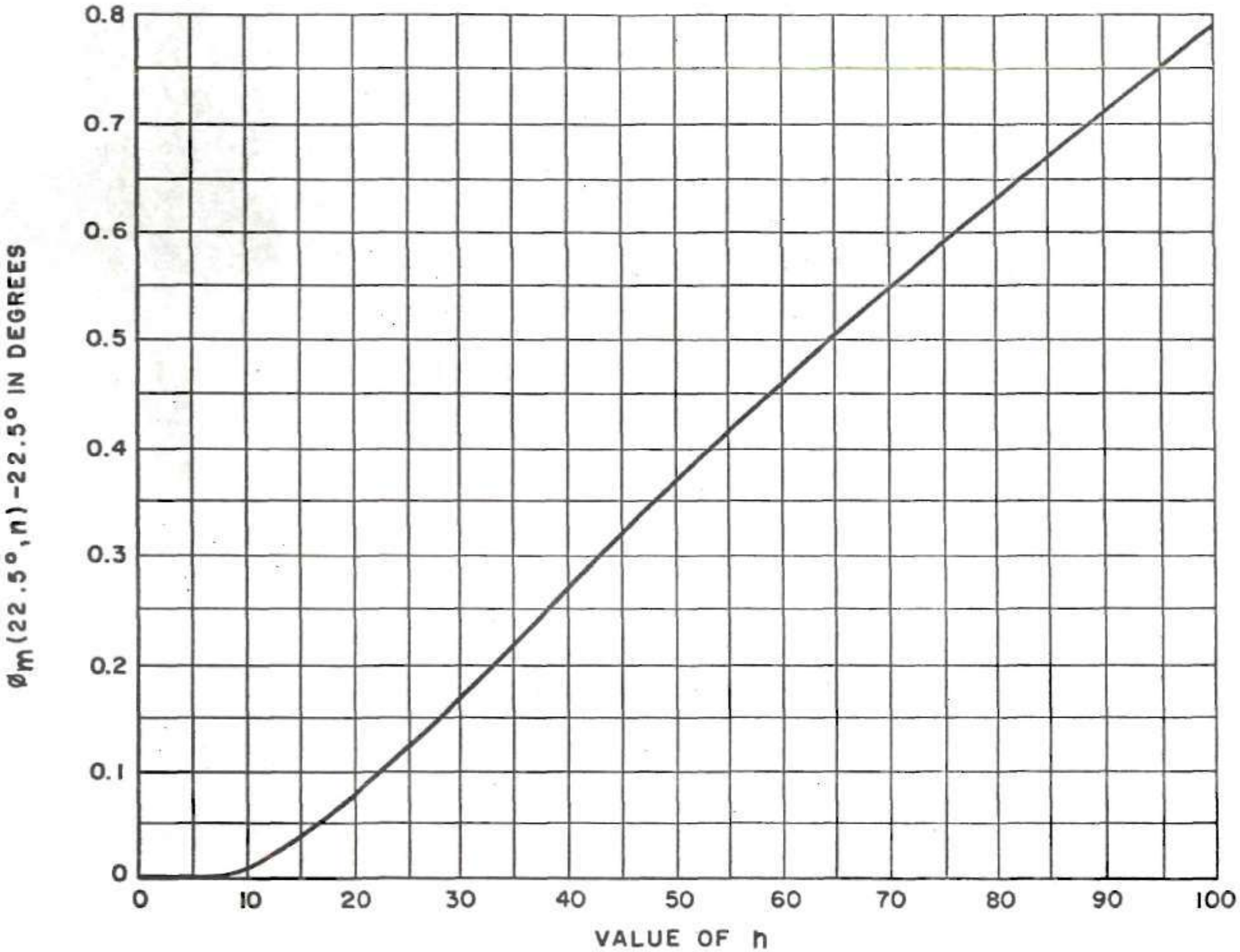


Fig. 9. Difference between ϕ_m and θ, for $\theta = 22.5°$.

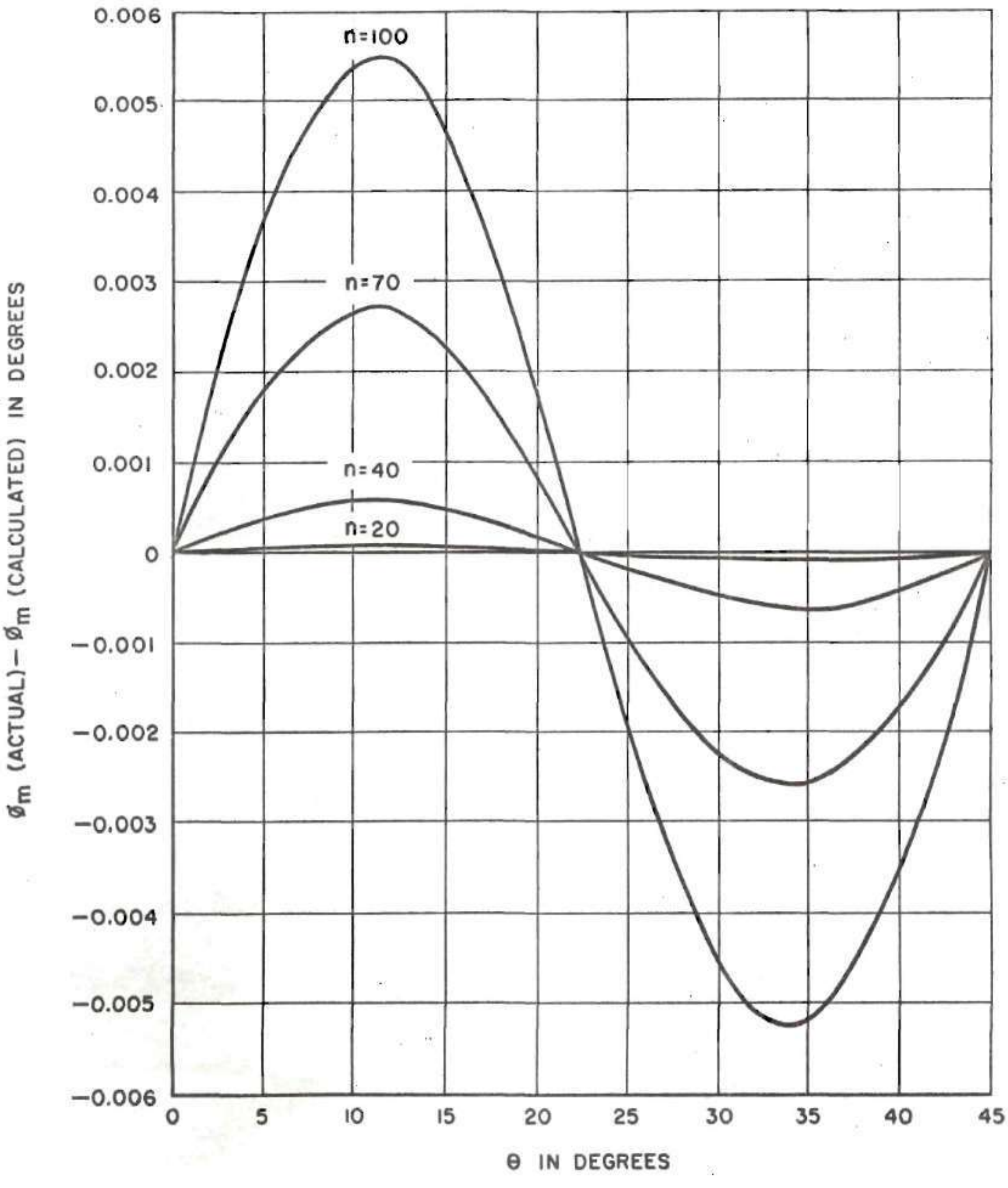


Fig. 10. Error in ϕ_m Caused by the Use of Equation 23.

CHAPTER IV

THE FINITE-BAND PHASE FUNCTION

The phase function for a practical network is derived from the infinite-band function of Chapter III by removing all phase sections whose phase characteristic centers lie outside the desired frequency band. This is partially justified by the fact that sections whose centers lie well outside the frequency band of interest have only a small effect inside the band. However, removal of sections which are centered near the band edges causes the phase characteristic to sag in that vicinity. This sag can be compensated for by replacing the two end sections with correction functions.

Tolerance Criteria for the Correction Function.--The problem of correcting for the undesirable effects caused by removing all phase sections lying outside a given frequency band may be approached most effectively by considering at first the removal of all sections lying on one side only. A correction function may be positioned to compensate for these removals. Then the effects of removing sections on the opposite side of the band can be corrected with a correction function identical in shape with the first, but shifted appropriately.

Fig. 11 shows the phase function α that results when all sections centered at frequencies less than n are removed. The purpose of the correction function is to shift the function of Fig. 11 back into the original tolerance band, ϕ_{min} to ϕ_{max}, as completely as possible. Thus, the difference between the limits of the original tolerance band and the curve in Fig. 11 defines an area in which as much of the correction function as possible should lie. This area (shaded) is shown in Fig. 12. Superimposed on this area is the phase function α', which is the phase angle of the pole-zero pairs that were removed. The curves for α and α' must be symmetrical with respect to the frequency $\omega = n^{1/2}$. The shaded area has a width of $\Delta\phi$ for all frequencies. For frequencies less than $n^{1/2}$ the lower limit of the shaded area follows a line described by $\phi_{min}-\alpha$. For frequencies greater than $n^{1/2}$ the shaded area ripples about a line which is the mirror image of α about $\omega = n^{1/2}$.

If, for $\omega \geq n^{1/2}$, the correction function could be made to coincide with α', the original phase function would be restored for this frequency range. Approximation of α' in this region will be, in fact, a necessity because of the non-rippling character of the correction function (given in the next section). If, for example, the correction function were greater than α' at $\omega = n^{5/2}$, then

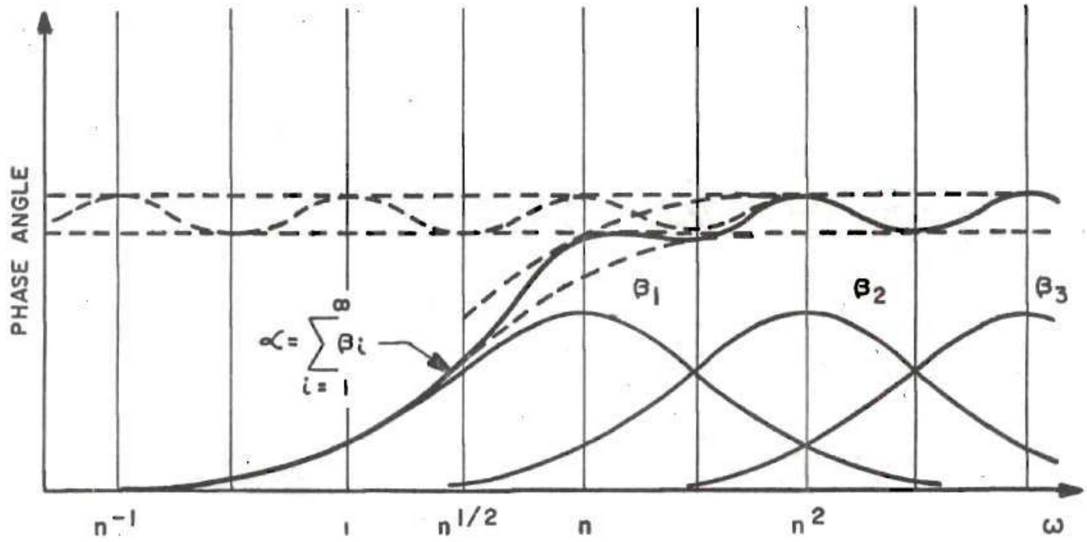


Fig. 11. The Effect of Removing End Sections.

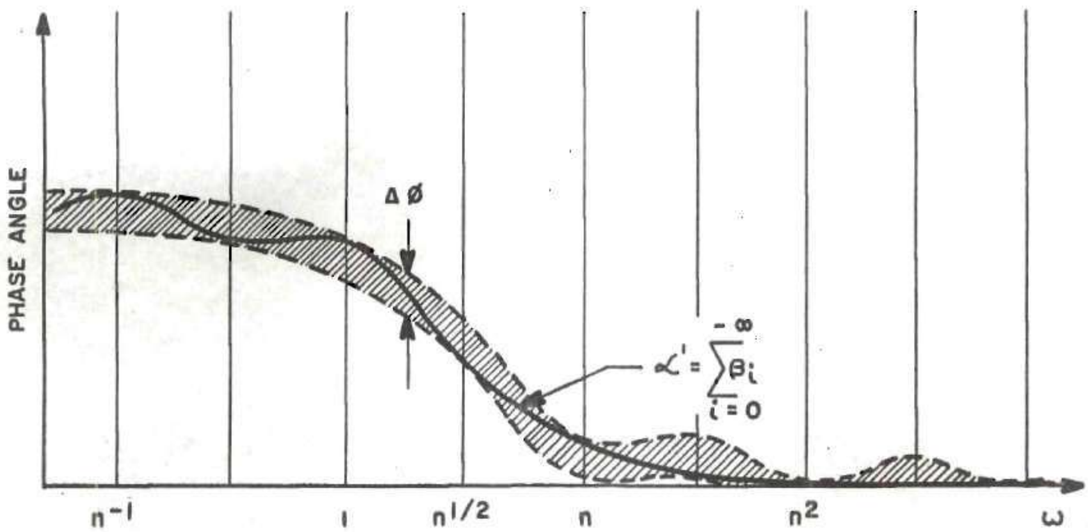


Fig. 12. Tolerance Band for the Correction Function.

it would certainly be greater than α' at $\omega = n^2$ or $\omega = n^3$ or both.

As a step toward the ultimate development of a reliable method of approximating α' for frequencies greater than or equal to $n^{1/2}$, analytical expressions are now developed to describe α' in this region. Values of $\alpha'(\omega = n^{i/2})$, where i is an integer, may be written by inspection. For $i = 1$ or $\omega = n^{1/2}$,

$$\alpha'(\omega = n^{1/2}) = \frac{\phi_{min}}{2} \tag{24}$$

For $i = 2$

$$\alpha'(\omega = n) = \frac{\phi_{max}}{2} - \frac{1}{2}\tan^{-1}\left[\frac{k^{1/2} - k^{-1/2}}{2}\right] \tag{25}$$

In general, for $i = 3,4,5,\ldots,$

$$\alpha'(\omega = n^{i/2}) = \alpha(\omega = n^{\frac{i-2}{2}}) - \tan^{-1}\left[\frac{k^{1/2} - k^{-1/2}}{n^{\frac{i-2}{2}} + \frac{1}{n^{\frac{i-2}{2}}}}\right] \tag{26}$$

It is possible, by making certain justifiable approximations, to derive a closed-form expression for α'. Consider the following exact expression for α'.

$$\alpha' = \sum_{i=0}^{-\infty} \tan^{-1}\left[\frac{A(k)}{\frac{\omega}{n^i} + \frac{n^i}{\omega}}\right] \tag{27}$$

where $\qquad A(k) = k^{1/2} - k^{-1/2}$

If $\omega >> n^i$ or if $\omega >> 1$, the above equation may be approximated by

$$\alpha' = \sum_{i=0}^{-\infty} \tan^{-1}\left[\frac{n^i\, A(k)}{\omega}\right]$$

or $$\alpha' = \tan^{-1}\frac{A(k)}{\omega} + \tan^{-1}\frac{A(k)}{\omega n} + \tan^{-1}\frac{A(k)}{\omega n^2} + \cdots$$

Combine the first two terms of the previous expression to obtain

$$\alpha' = \tan^{-1}\left[\frac{\frac{A(k)}{\omega}\,(1+\frac{1}{n})}{1-\left(\frac{A(k)}{\omega}\right)^2 \frac{1}{n}}\right] + \tan^{-1}\frac{A(k)}{\omega n^2} + \cdots$$

For $\omega >> A(k)/n^{1/2}$, the above expression may be approximated by

$$\alpha' = \tan^{-1}\left[\frac{A(k)}{\omega}(1+\frac{1}{n})\right] + \tan^{-1}\frac{A(k)}{\omega n^2} + \tan^{-1}\frac{A(k)}{\omega n^3} + \cdots$$

Since $$\frac{A(k)}{n^{1/2}} < \sqrt{\frac{k}{n}} < 1$$

$\omega >> 1$ is the stronger of the two previous approximation conditions. Repeat the above procedure, starting with the combination of the first two terms in the last equation. The result is

$$\alpha' = \tan^{-1}\left[\frac{A(k)}{\omega}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)\right] + \tan^{-1}\frac{A(k)}{\omega n^3} + \cdots$$

for $\quad \omega^2 >> \left(\frac{A(k)}{n^{1/2}}\right)^2\left(1+\frac{1}{n}\right)\frac{1}{n}$

At this point a trend is recognizable in the condition necessary to allow simplification of the first term and also in the term resulting from the simplification. In general, the approximation condition is

$$\omega^2 >> \left(\frac{A(k)}{n^{1/2}}\right)\frac{\left(1+\frac{1}{n}+\cdots+\frac{1}{n^i}\right)}{n^i}$$

where $\quad i = 0,1,2,\ldots$

For most practical purposes and as far as this paper is concerned, n will always be greater than two. With this restriction on n, it is apparent that

$$\frac{1+\frac{1}{n}+\cdots+\frac{1}{n^i}}{n^i} \leq 1$$

And as shown previously,

$$\frac{A(k)}{n^{1/2}} < 1$$

Thus, it is seen that the strongest approximation condition is still $\omega \gg 1$.

In the limit, α' becomes

$$\alpha' = \tan^{-1}\left[\frac{A(k)}{\omega}\left(1+\frac{1}{n}+\frac{1}{n^2}+\cdots\right)\right] \tag{28}$$

which may be written

$$\alpha' = \tan^{-1}\frac{nA(k)}{\omega(n-1)} \tag{29}$$

The Correction Function.--The most obvious correction function would consist of the phase function of one or more pole-zero pairs appropriately shifted and adjusted in magnitude. The larger the number of pole-zero pairs used, the more tolerance band could be accommodated. It is proposed to consider here only one pole-zero pair as providing the correction function.

Let the correction function be defined by the following equation.

$$\beta_c = \tan^{-1}\left[\frac{A(k_c)}{\frac{\omega}{\ell_c}+\frac{\ell_c}{\omega}}\right] \tag{30}$$

where $$A(k_c) = k_c^{1/2} - k_c^{-1/2}$$

And, with reference to the frequency scale in Fig. 12, ℓ_c designates the center frequency of this function. Since equation 30 contains two unknowns or two degrees of freedom, two relationships between the previously described tolerance band and β_c are required to completely define the correction function. To obtain the first relationship equate α' and β_c for large ω. For $\omega >> \ell_c$ equation 30 becomes

$$\beta_c = \tan^{-1} \frac{\ell_c \, A(k_c)}{\omega} \tag{31}$$

Equating equations 29 and 31, one has

$$\tan^{-1} \frac{nA(k)}{\omega(n-1)} = \tan^{-1} \frac{\ell_c \, A(k_c)}{\omega}$$

or

$$\ell_c \, A(k_c) = \frac{nA(k)}{n-1} \tag{32}$$

The second relationship may be determined by forcing β_c to coincide with some point in the tolerance band where the approximations, $\omega >> \ell_c$ and $\omega >> 1$, are invalid. Two different points on α', given by equations 24 and 25, will be considered here. Subscripts 1 and 2 will be used to designate the results obtained by using equations 24 and 25, respectively.

Equate α' and β_c as given by equations 24 and 30, and let $\omega = n^{1/2}$ in equation 30.

$$\tan^{-1} \left[\frac{A(k_1)}{\frac{n^{1/2}}{\ell_1} + \frac{\ell_1}{n^{1/2}}} \right] = \frac{\phi_{min}}{2}$$

Taking the tangent of both sides and rearranging terms, one has

$$\ell_1^2 = \frac{\ell_1 \; n^{1/2} \; A(k_1)}{\tan \frac{\phi_{min}}{2}} - n$$

Substitute from equation 32 for $\ell_1 A(k_1)$ and solve for ℓ_1. The result is

$$\ell_1 = \sqrt{n\left[\frac{A(k) \; n^{1/2}}{(n-1)\tan \frac{\phi_{min}}{2}} - 1\right]} \tag{33}$$

From equation 32,

$$A(k_1) = \frac{A(k) \; n}{\ell_1 (n-1)} \tag{34}$$

Equations 33 and 34 define one possible correction function.

Performing the above operations with equation 25 in place of equation 24, one has the following defining equations for the second correction function:

$$\ell_2 = n \sqrt{\frac{A(k)}{(n-1) \; \tan \alpha' \; (\omega = n)} - 1} \tag{35}$$

or

$$\ell_2 = n \sqrt{\frac{A(k)}{(n-1)\tan\left[\frac{\phi_{max}}{2} - \frac{1}{2}\tan^{-1}\frac{A(k)}{2}\right]} -1} \tag{36}$$

and

$$A(k_2) = \frac{A(k) \; n}{\ell_2 (n-1)} \tag{37}$$

It is necessary to know the extent of the correction furnished by these two correction functions. A comprehensive set of curves which illustrate the behavior of the correction function with respect to the tolerance band are given in Figs. 13 and 14 for β_1 and in Figs. 15 and 16 for β_2. The frequency scales are the same as the one in Fig. 12. The ordinate in each case is the normalized difference between the correction function and α''. α'' is equal to α' for $\omega \geq n^{1/2}$ and is the lower limit of the tolerance band in Fig. 12 for $\omega \leq n^{1/2}$. Normalization with respect to $\Delta\phi$ is used so that one tolerance band applies to all cases. This universal tolerance band extends from zero to one for $\omega \leq n^{1/2}$. For $\omega > n^{1/2}$ the tolerance band, always of width one, ripples about the zero line just as the tolerance band in Fig. 12 ripples about the line α'. The tolerance band limits in this last region apply only to the phase curves plotted there, which have $n \leq 7$. For $n > 7$ the rippling part of the tolerance band deviates somewhat from that shown. However, the band shown is sufficient since for $n > 7$ the phase curves are essentially coincident with the zero line.

It should be kept in mind that these figures illustrate the results of correction on only one side of a given frequency band. Correction on the opposite side of the band will result in curves symmetrical with these

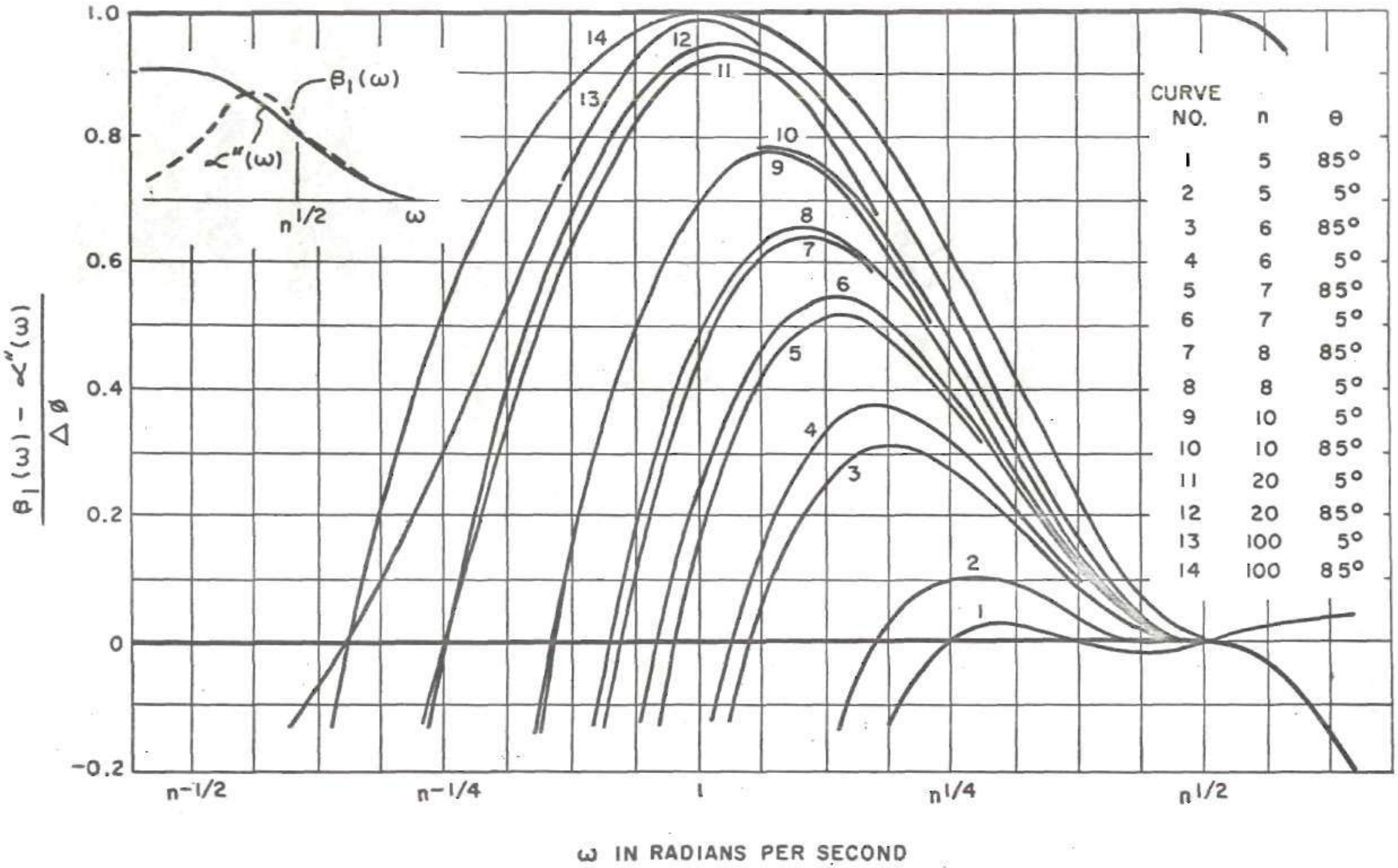


Fig. 13. The Correction Function β_1 Relative to Its Tolerance Band ($\omega \leq n^{1/2}$).

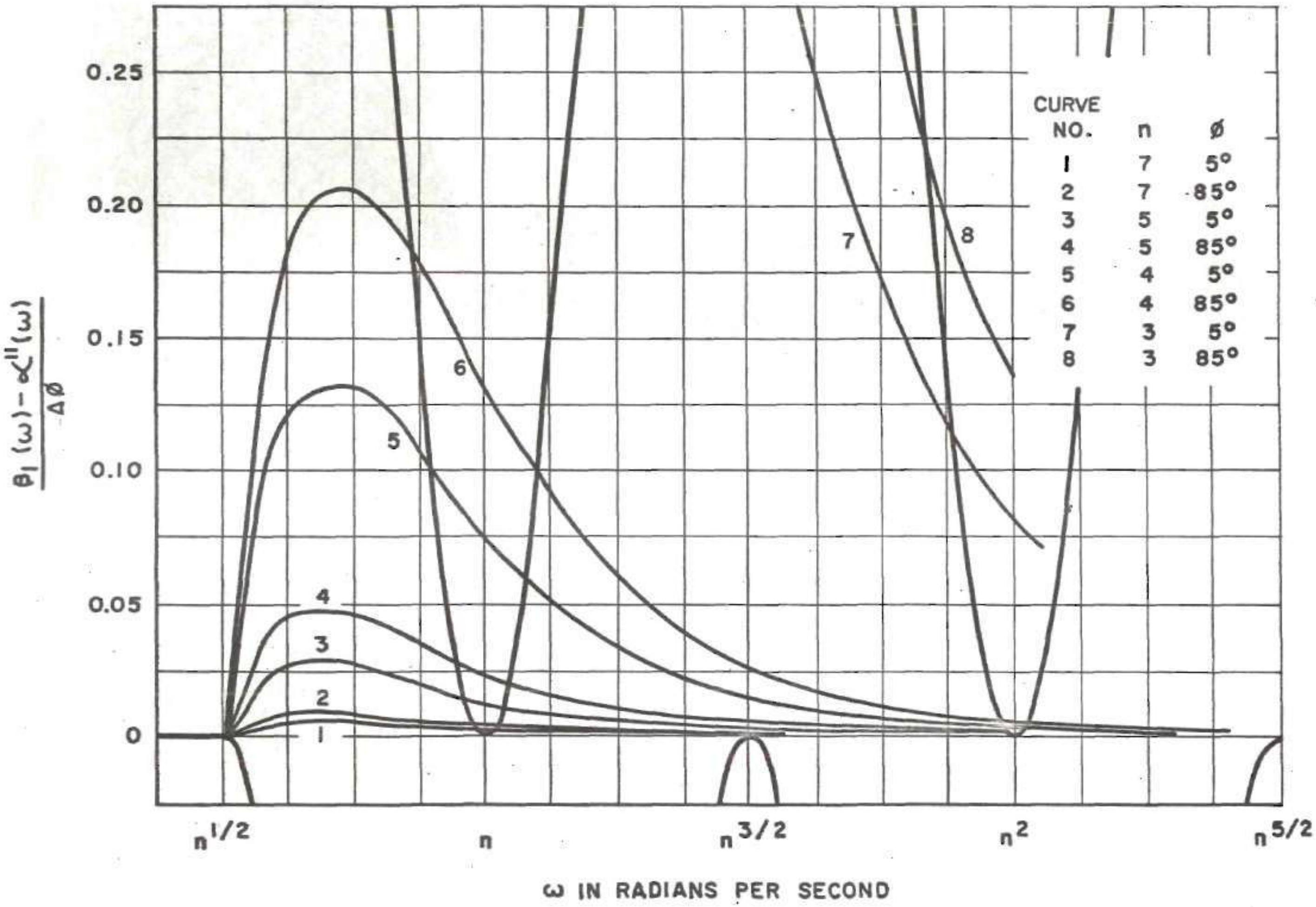


Fig. 14. The Correction Function β_1 Relative to Its Tolerance Band $(\omega \geq n^{1/2})$.

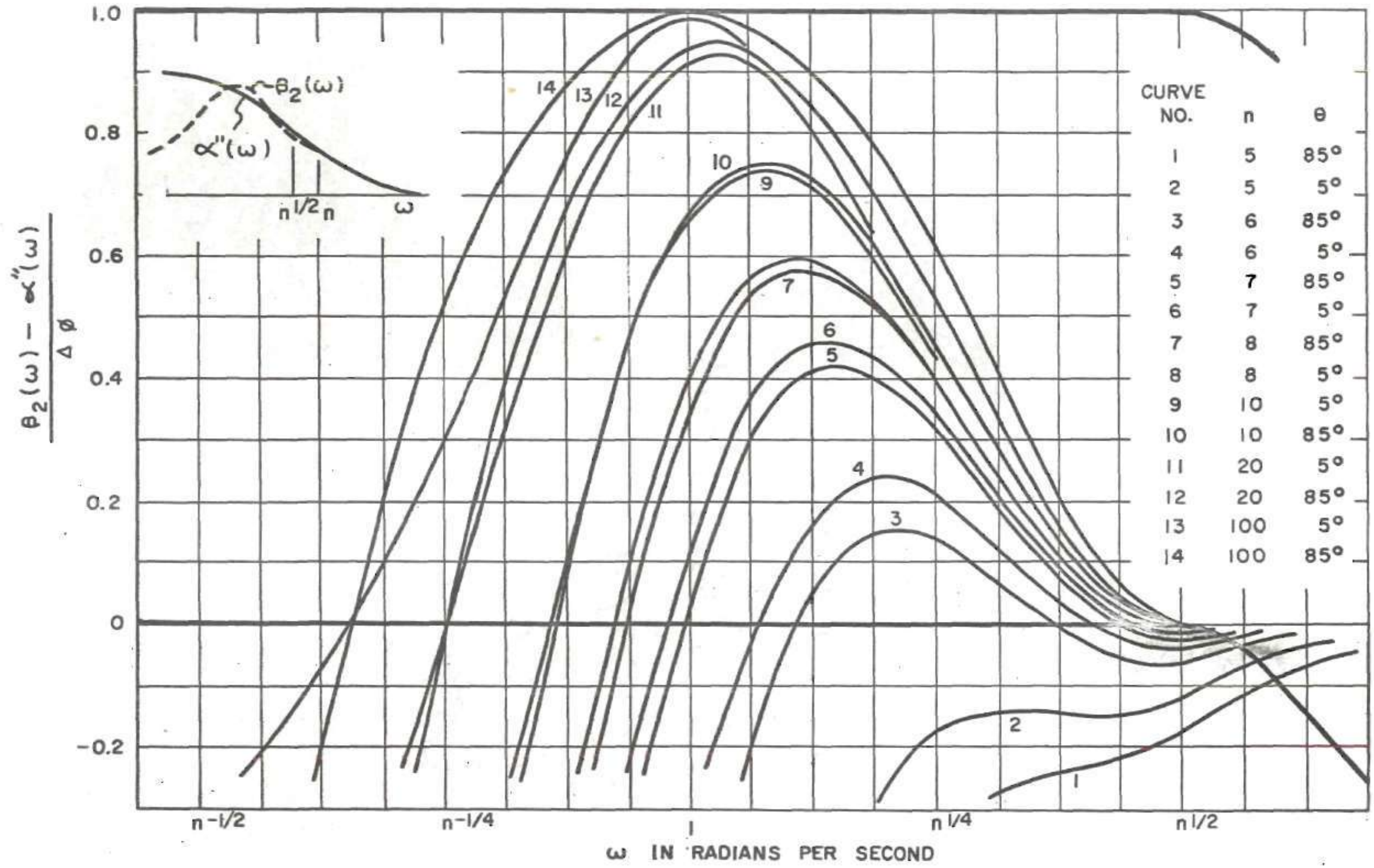


Fig. 15. The Correction Function β_2 Relative to Its Tolerance Band ($\omega \leq n^{1/2}$).

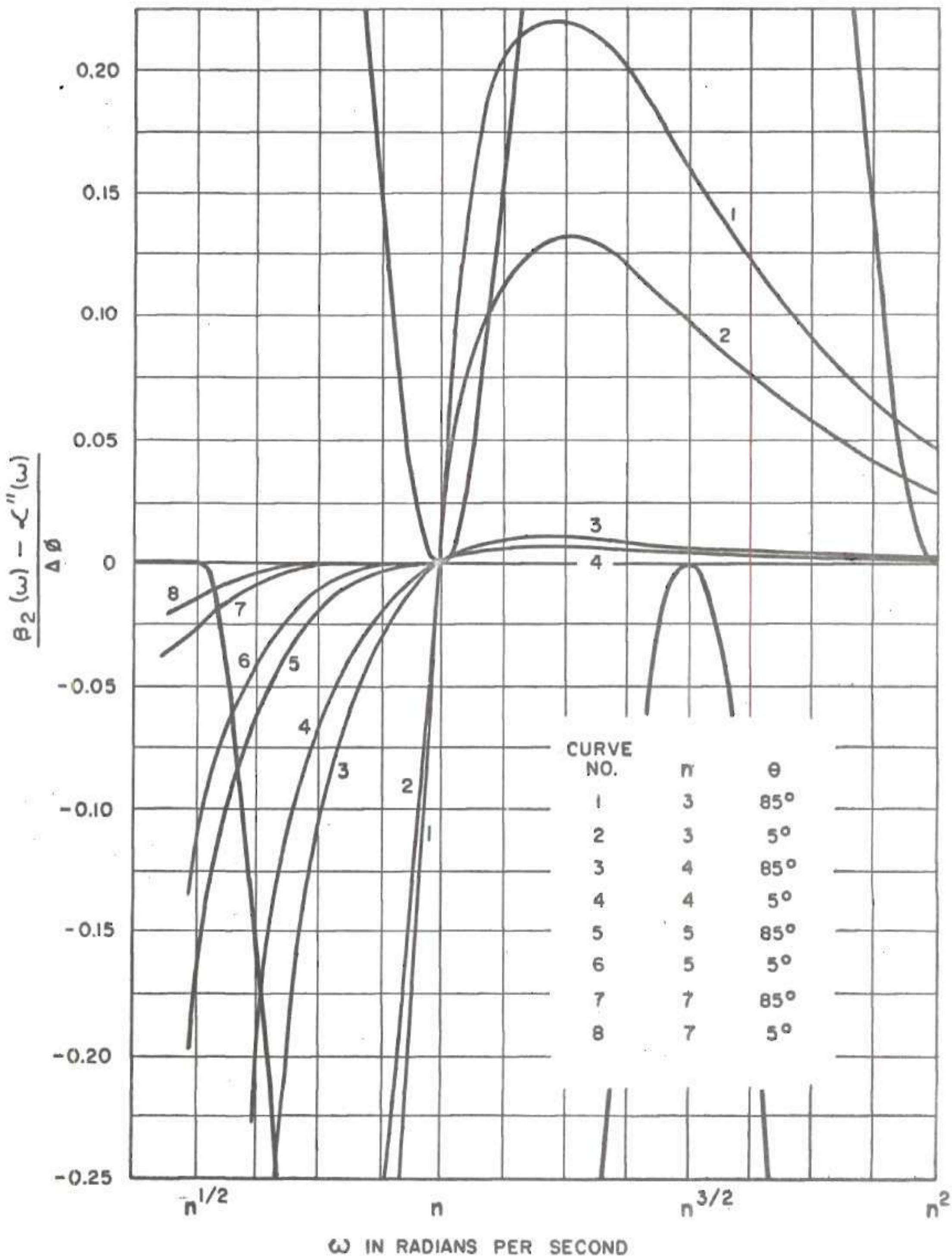


Fig. 16. The Correction Function β_2 Relative to Its Tolerance Band ($\omega \geq n^{1/2}$).

about the center of the band. If a particular curve is nonzero at and beyond the center of the band, then overlapping occurs; and the combined effect of both correction functions must be considered. For example, consider a case where the final phase function consists of just two correction functions for which $n = 5$ and $\theta = 85°$. With reference to Figs. 13 and 14, the center of the frequency band would be at $\omega = n^{1/2}$. To find the overall results, the mirror image (with respect to $\omega = n^{1/2}$) of the curve in Fig. 14 must be added to the corresponding curve in Fig. 13.

For $n \geq 7$ the two correction functions give comparable results. β_1 is preferred because ℓ_1 is easier to calculate than ℓ_2. As n increases, the maximum point in Figs. 13 and 15 approaches 1.0 at $\omega = 1.0$. This is as expected, since for large n the line α' for $\omega \geq n^{1/2}$ is, for practical purposes, just a part of one of the phase sections that were removed. Thus, the correction function would tend to be identical in shape and position with the first section that was removed.

For $n < 7$ use of β_1 will result in a wider bandwidth, but may give prohibitive deviation from the tolerance band at $\omega = n$. In particular, for $n = 7$ a maximum deviation of $2(0.0026)\Delta\phi$ or $0.0052\Delta\phi$ may occur. For $n = 5$ and $n = 4$ the maximum errors possible are $0.05\Delta\phi$ and $0.28\Delta\phi$, respectively. Use of β_2 gives less error for $\omega \geq n$. However, as n decreases, deviation from the band near $\omega = n$ increases regardless of which correction function is used.

Bandwidth.--With the development of the data in the previous section, a complete discussion of bandwidth is possible.

Consider a series of r identical phase sections. The ratio of the center frequencies of the two end sections may be taken as a first approximation to the bandwidth which results when the two end sections are replaced by correction functions. The mathematical expression is

$$B \approx n^{r-1} \tag{38}$$

The extent of correction furnished by the correction functions will determine whether the bandwidth is greater of less than that given by equation 38. An exact expression for bandwidth may be written as

$$B = \frac{n^{r-1}}{(n^a)^2} \tag{39}$$

or

$$B = n^{r-1-2a} \tag{40}$$

$$= \frac{\text{Upper Frequency Limit}}{\text{Lower Frequency Limit}}$$

The factor n^a in equation 39 may be found from Fig. 13, 14, 15, or 16 as the frequency at which the function first enters the tolerance band. For example, if $n = 5$ and $\theta = 85°$, then $n^a = n^{1/4}$ from Fig. 13.

A graphical means is convenient for predicting, at the beginning of a design problem, the bandwidth attainable for a given phase tolerance and a given network complexity. However, a family of curves relating $\Delta\phi$ to B with r as the family parameter would be too cluttered to be useful because $\Delta\phi$ is a function of both θ and n. For a given bandwidth and a given network complexity, $\Delta\phi$ is not unique. This difficulty may be circumvented by using $\Delta\phi(45°,n)$ instead of $\Delta\phi(\theta,n)$. $\Delta\phi(45°,n)$ can easily be calculated from $\Delta\phi(\theta,n)$ and ϕ_m with the aid of equation 22. Although a in equation 40 is a function of both θ and n, its variation with θ is small, as shown in Figs. 13, 14, 15 and 16.

Equation 40 was used in conjunction with Table 5 ($\Delta\phi(45°,n)$ versus n) to obtain Fig. 17 which relates $\frac{\Delta\phi(45°)}{2}$, B, and r. For each value of n, the maximum value of n^a was used. Both correction functions are represented. The curves which represent β_1 are for $n \geq 4$, and those for β_2 are for $n \leq 5$.

All possible networks are represented by the given curves. Thus, in general it will be impossible to realize both phase tolerance and bandwidth exactly as desired. A favorable change in one of the three variables B, $\Delta\phi$, or r must be accompanied by an adverse change in one or both of the remaining two.

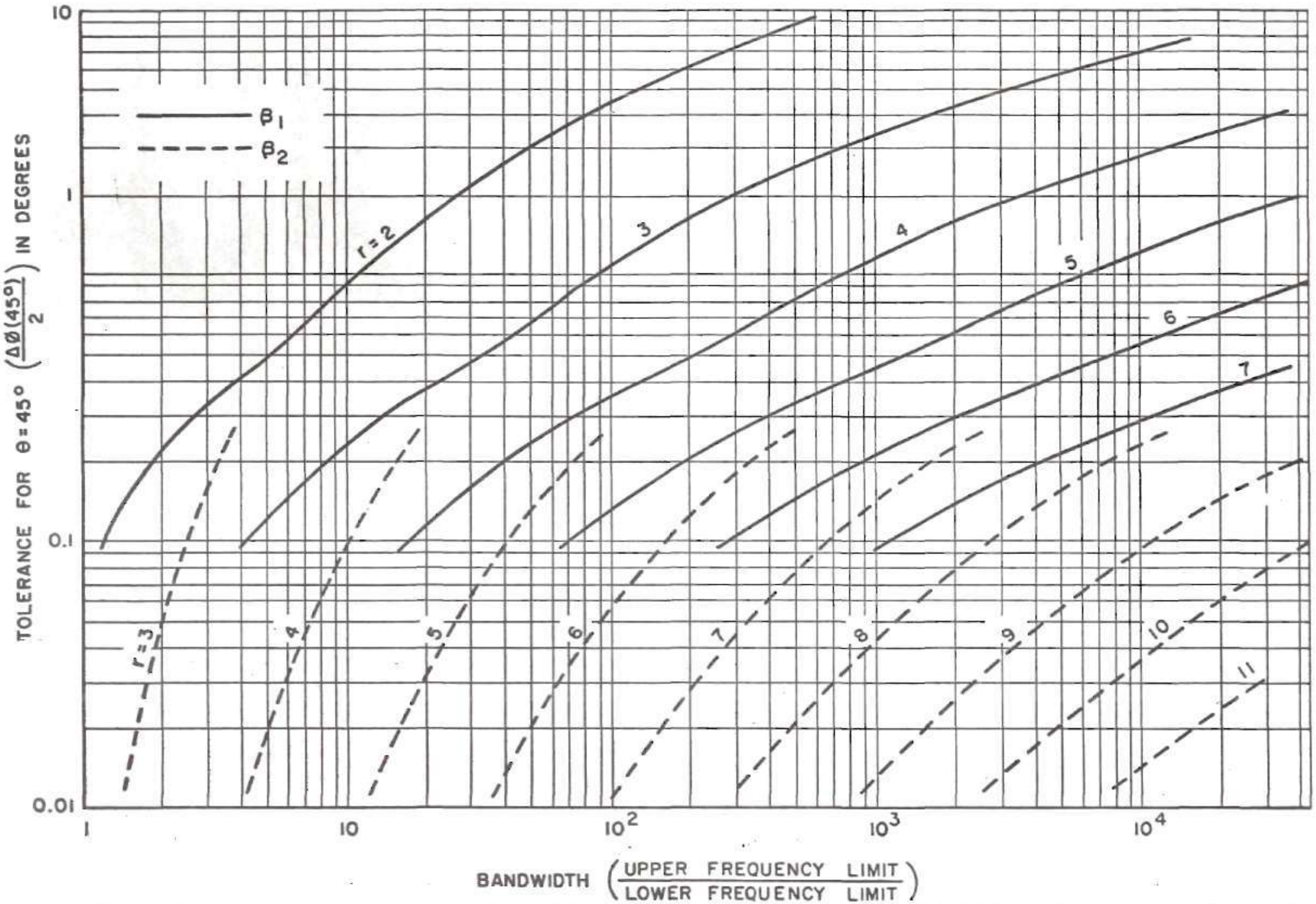


Fig. 17. Tolerance as a Function of Bandwidth with Various Degrees of Network Complexity, for $\theta = 45°$.

CHAPTER V

ALTERNATE NETWORKS AND INTERPRETATIONS

The preceding investigation has been based on the use of standard RC network sections cascaded with ideal amplifiers as a practical means of realizing a desired voltage ratio. This arrangement would be satisfactory in a case where the amplifiers were necessary for reasons other than simply to separate constant-phase network sections. In other situations it may be possible to synthesize a more suitable network from the resulting pole-zero configuration.

Further flexibility may be attained by placing different interpretations on the pole-zero plot. For some applications it might prove beneficial to interpret the pole-zero plot as describing a transfer impedance, a transfer admittance, or a current ratio. Also, because poles and zeros alternate on the negative real frequency axis, the pole-zero plot may be interpreted as representing an RC or RL driving point immittance.

A few transfer function synthesis methods are discussed briefly in the following sections.

RC Networks.--For many low frequency applications, networks containing only resistors and capacitors are required. A variety of methods are found in the literature for both lattice and unbalanced networks.

Bower and Ordung (8) present a lattice network realization for voltage-ratio functions, for both open-circuit and RC loaded output terminal-pairs. Methods of unbalancing the lattice are outlined.

A lattice network realization developed by Orchard (9) is particularly simple and straightforward. The method results in a single RC lattice with a removable series resistance in each arm. The network is symmetrical and requires a generator and load having equal resistive impedances. One disadvantage is that one transformer is required if the lattice must be unbalanced.

Fleck and Ordung (10) offer a method for the exact realization of a transfer ratio with a ladder network. This requires the synthesis of an approximate network, which is then used as a guide to the synthesis of the exact network. It is advantageous in that it allows the maximum value of gain to be realized.

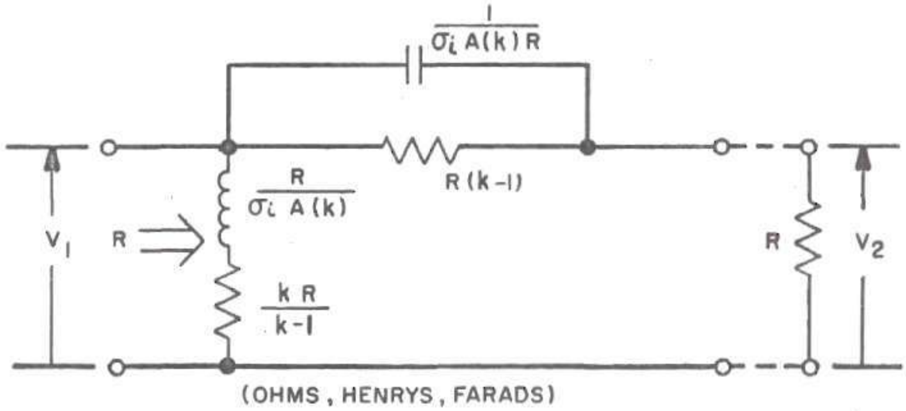
E. A. Guillemin (11) discusses a simple zero shifting technique for realizing a transfer function as a ladder network. This method applies only to a transfer function which has all its zeros on the negative real frequency axis.

Other methods are given by Dasher (12), Fialkow and Gerst (13), and Ordung, Hopkins, Krauss, and Sparrow (14).

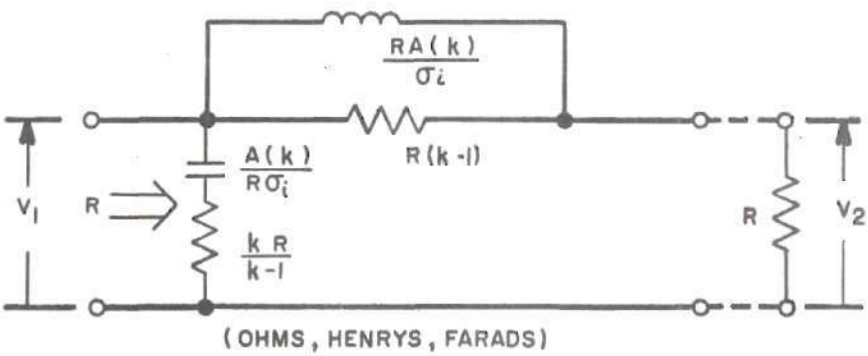
RLC Networks.--For applications where all three element types may be included in a circuit, a practical and simple network may be built up by cascading constant-resistance sections. Typical sections for positive and negative phase angles are given in Fig. 18. These networks were synthesized by a method described in detail by Balabanian (15). Note that the transfer function is independent of R. The parameter R must be fixed by external circuit requirements such as the load on a vacuum tube. One section is required for each pole-zero pair. A comparison of these network sections with the section of Figs. 1 and 2 shows that the need for isolating amplifiers has been eliminated at the expense of just two extra elements per section.

Alternate constant-resistance ladder sections are given in a paper by Bresler (16).

Guillemin (17) discusses a method for synthesizing a constant resistance lattice. For pole-zero plots encountered in this thesis, the lattice may always be unbalanced. The result is a symmetrical network which has the same configuration as one of the networks given in the previously mentioned paper by Bresler.



$$\frac{V_2}{V_1} = \frac{s + \sigma_i \ k^{-1/2}}{s + \sigma_i \ k^{1/2}}$$



$$\frac{V_2}{V_1} = \frac{1}{k} \left(\frac{s + \sigma_i \ k^{1/2}}{s + \sigma_i \ k^{-1/2}} \right)$$

Fig. 18. Constant-Resistance Ladder Network Sections.

CHAPTER VI

SUMMARY OF DESIGN PROCEDURE AND A DESIGN EXAMPLE

Design Procedure.--Given:

ϕ_m--the "average" phase angle $\left(\frac{\phi_{max} + \phi_{min}}{2}\right)$

$\Delta\phi(\theta,n)/2$--the phase tolerance $\left(\frac{\phi_{max} - \phi_{min}}{2}\right)$

B--the ratio of upper and lower frequency limits

1. Calculate $\Delta\phi(45°,n)$ using equation 22.

$$\Delta\phi(45°,n) = \frac{\Delta\phi(\theta,n)}{\sin 2\phi_m}$$

2. Use Fig. 17 to decide on final values of $\Delta\phi$, B, and r.
3. Find n from Fig. 8 or Table 5.
4. Find $\phi_m(22.5°,n)-22.5°$ from Fig. 9 or Table 6.
5. Use equation 23 to calculate θ by successive approximations.

$$\theta = \phi_m(\theta,n) - \left[\phi_m(22.5°,n)-22.5°\right] \sin 4\theta$$

For the first approximation use $\sin 4\phi_m$ instead of $\sin 4\theta$.

6. Determine $k^{1/2}$ from equation 9, and then calculate A(k).

$$k^{1/2} = n^{\theta/180}$$

$$A(k) = k^{1/2} - k^{-1/2}$$

7. Find $\tan \alpha'(\omega=n^{1/2})$ or $\tan \alpha'(\omega=n)$ with the aid of equation 24 or equation 25, respectively.

For β_1 $\quad \tan \alpha'(\omega = n^{1/2}) = \tan \frac{\phi_{min}}{2}$

For β_2 $\quad \tan \alpha'(\omega = n) = \tan \left(\frac{\phi_{max}}{2} - \frac{1}{2} \tan^{-1} \frac{A(k)}{2} \right)$

8. Calculate ℓ_c and $A(k_c)$ from equations 33 and 34 or equations 36 and 37.

For β_1
$$\begin{cases} \ell_1 = \sqrt{n \left[\dfrac{A(k) n^{1/2}}{(n-1) \tan \frac{\phi_{min}}{2}} - 1 \right]} \\[2ex] A(k_1) = \dfrac{A(k) n}{\ell_1 (n-1)} \end{cases}$$

For β_2
$$\begin{cases} \ell_2 = n \sqrt{\dfrac{A(k)}{(n-1) \tan \left[\frac{\phi_{max}}{2} - \frac{1}{2} \tan^{-1} \frac{A(k)}{2} \right]} - 1} \\[2ex] A(k_2) = \dfrac{A(k) n}{\ell_2 (n-1)} \end{cases}$$

9. Calculate $k_c^{1/2}$ from

$$k_c^{1/2} = \frac{A(k_c)}{2} + \sqrt{\left[\frac{A(k_c)}{2} \right]^2 + 1}$$

10. Determine the transfer function so that the frequency band is centered at $\omega = 1$.

$$\frac{V_2}{V_1} = \left(\frac{s + \ell_c k_c^{-1/2} \sigma_1}{s + \ell_c k_c^{1/2} \sigma_1} \right) \left(\frac{s + k^{-1/2} \sigma_2}{s + k^{1/2} \sigma_2} \right) \cdots \left(\frac{s + k^{-1/2} \sigma_{r-1}}{s + k^{1/2} \sigma_{r-1}} \right) \left(\frac{s + \ell_c^{-1} k_c^{-1/2} \sigma_r}{s + \ell_c^{-1} k_c^{1/2} \sigma_r} \right)$$

$$\sigma_i = n^{\frac{2i-r-1}{2}}$$

and $$\sigma_{r-i+1} = \frac{1}{\sigma_i}$$

where $i = 1,2,3,\ldots,r$

11. Determine the elements of a network. Element values in terms of σ_i and k are given in Fig. 19 for the basic networks of Chapter II. One network section is required for each bracketed term of the voltage ratio in part 10. To find the networks for the end sections, substitute $\sigma_i \ell_c$ for σ_i and k_c for k in Fig. 19.

For other networks see Chapter V.

Example.--Given: $\phi_m = 55°$

$\Delta\phi(\theta,n) = 2°$

B = 2 Decades

1. $\Delta\phi(45°,n) = \frac{2°}{\sin(2x55°)} = 2.1284$
2. From Fig. 17, r=3. No adjustment in the design parameters is necessary.
3. From Table 5 (by interpolation), n=8.2426.
4. From Table 6 (by interpolation),

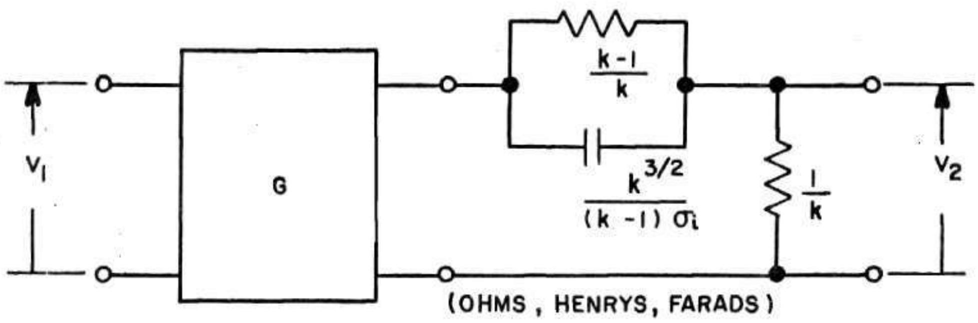
   $$\phi_m(22.5°,n)-22.5° = 0.0049°$$
5. First approximation:

   $$\theta \approx 55° - (0.0049°)\sin(4x55°) = 55.0032°$$

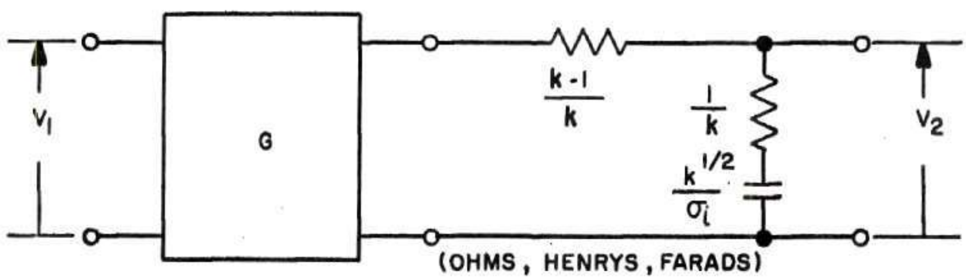
   Second approximation:

   $$\theta \approx 55° - (0.0049°)\sin(4x55.0032°) = 55.0032°$$

   Therefore, $\theta = 55.0032°$



$$\frac{V_2}{V_1} = G\left(\frac{s + \sigma_i\, k^{-1/2}}{s + \sigma_i\, k^{1/2}}\right)$$



$$\frac{V_2}{V_1} = \frac{G}{k}\left(\frac{s + \sigma_i\, k^{1/2}}{s + \sigma_i\, k^{-1/2}}\right)$$

Fig. 19. Basic Network Sections with Normalized Element Values.

6. $k^{1/2} = (8.2426)^{\frac{55.0032}{180}} = 1.9052$

$A(k) = 1.9052 - \frac{1}{1.9052} = 1.3803$

7. $\tan \frac{54°}{2} = 0.50953$

8. $\ell_1 = \sqrt{[8.2426]\left[\frac{(1.3803)(8.2426)^{1/2}}{(7.2426)(0.50953)} - 1\right]}$

$\ell_1 = 0.7801$

$A(k_1) = \frac{(1.3803)(8.2426)}{(0.7801)(7.2426)}$

$A(k_1) = 2.0137$

9. $k_1^{1/2} = \frac{2.0137}{2} + \sqrt{\left(\frac{2.0137}{2}\right)^2 + 1}$

$k_1^{1/2} = 2.4259$

10. $\sigma_1 = (8.2426)^{\frac{2-3-1}{2}} = 0.12132$

$\sigma_2 = (8.2426)^{\frac{4-3-1}{2}} = 1$

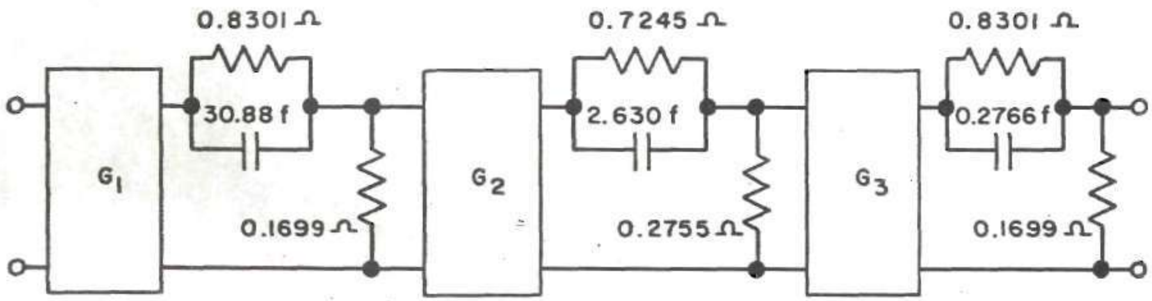
$\sigma_3 = 1/\sigma_1 = 8.2426$

$\sigma_1\ell_1 = (0.7801)(0.12132) = 0.09464$

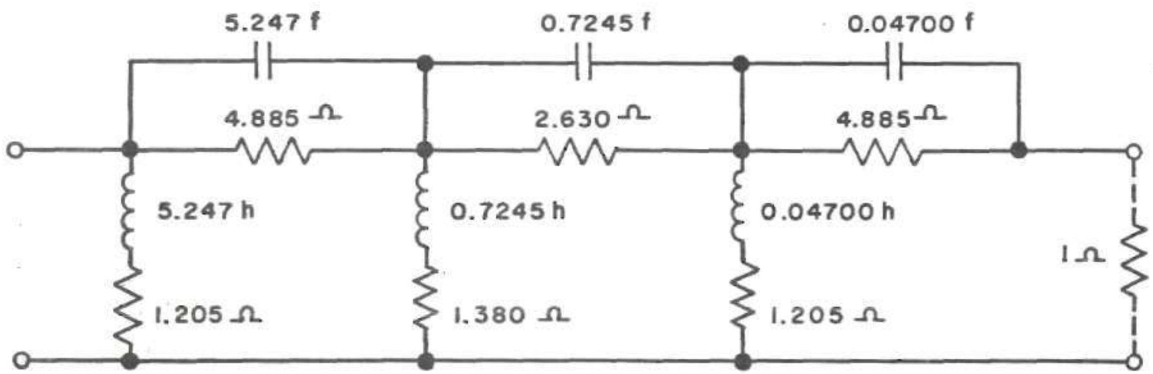
$\sigma_3\ell_1^{-1} = 1/\sigma_1\ell_1 = 10.57$

$$\frac{V_2}{V_1} = \left[\frac{s + \frac{0.09464}{2.4259}}{s + (2.4259)(0.09464)}\right]\left[\frac{s + \frac{1}{1.9052}}{s + 1.9052}\right]\left[\frac{s + \frac{10.57}{2.4259}}{s + (2.4259)(10.57)}\right]$$

11. The networks in Fig. 20 are determined in accordance with Figs. 18 and 19.



NETWORK UTILIZING BASIC NETWORK SECTIONS



NETWORK UTILIZING CONSTANT-RESISTANCE LADDER SECTIONS (R=1)

Fig. 20. Example Networks for $\phi_m = 55°$, $\Delta\phi = 2°$, and B = 2 Decades.

To conclude the design example, a few points on the actual phase curve are calculated below.

$$\phi(\omega) = \tan^{-1}\left[\frac{A(k_1)}{\frac{\omega}{\ell_1\sigma_1} + \frac{\ell_1\sigma_1}{\omega}}\right] + \tan^{-1}\left[\frac{A(k)}{\omega + \frac{1}{\omega}}\right]$$

$$+ \tan^{-1}\left[\frac{A(k_1)}{\frac{\omega}{\ell_1^{-1}\sigma_3} + \frac{\ell^{-1}\sigma_3}{\omega}}\right]$$

$$\phi(\omega) = \tan^{-1}\left[\frac{2.0137}{\frac{\omega}{0.09464} + \frac{0.09464}{\omega}}\right] + \tan^{-1}\left[\frac{1.3803}{\omega + \frac{1}{\omega}}\right]$$

$$+ \tan^{-1}\left[\frac{2.0137}{\frac{\omega}{10.57} + \frac{10.57}{\omega}}\right]$$

For $\omega = 1$, $\phi = 56.003°$ (maximum point at center of band).
For $\omega = n^{1/2}$, $\phi = 54.000°$ (minimum point).
For $\omega = n^{0.9}$, $\phi = 55.350°$ (maximum point nearest band edge).
For $\omega = 10$, $\phi = 54.029°$ (band edge).

A 0.003° deviation from the tolerance band occurs at $\omega = 1$. All or most of this deviation may be attributed to the correction functions, as indicated by an extrapolation of the data in table 10. Any additional error is probably due to the limited accuracy imposed by the use of five-place logarithm tables. The most critical point was the calculation of ℓ_1, where a comparatively small difference between two large numbers was encountered.

CHAPTER VII

SUMMARY AND CONCLUSIONS

A design procedure for the realization of a network whose transfer function possesses a nearly-constant phase angle has been developed. Any phase angle from -90° to +90° is realizable. Detailed information for tolerances as low as ±0.015° for a phase angle of 45° is given. Smaller tolerance designs are possible for angles other than 45°.

For a fixed phase angle and a fixed phase tolerance, only discrete bandwidths are possible; and these depend on network complexity.

The design method is easy to apply by virtue of its mathematically simple, non trial and error procedure. Network complexity is not a limiting factor because the method results in the specification of pole-zero pairs, each of which may be realized with a simple network section connected in cascade.

On possible disadvantage is that two special tables (or two graphs) are required. Their use is necessary because of the lack of a closed-form expression for an infinite summation, which is used to attain the specified phase angle and phase tolerance.

The accuracy that may be expected depends on certain approximations made to describe the previously mentioned infinite summation and on a so-called correction procedure. All inaccuracies caused by approximations are such as to result in a smaller phase tolerance. For tolerances greater than $\pm 0.1°$, the maximum possible excursion (caused by the correction procedure) from the tolerance band is one per cent of the tolerance. Larger deviations may occur near the edges of the frequency band for tolerances less than $\pm 0.1°$.

APPENDIX I

PROOF OF PERIODICITY OF $\emptyset$

As given by equation 5,

$$\emptyset(k,n,\omega) = \sum_{i=-\infty}^{\infty} \tan^{-1} \left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega}{n^i} + \frac{n^i}{\omega}} \right)$$

It is proposed to show that $\emptyset$ is periodic on a logarithmic frequency scale. Since identical phase sections are spaced at intervals of log n, as shown in Fig. 3 (Chapter III), a period of log n is anticipated. To test this conjecture, compare $\emptyset$ at $\omega = \omega'$ with $\emptyset$ at $\omega = n\omega'$, which is a distance log n from ω'. The two functions are respectively,

$$\emptyset(k,n,\omega') = \sum_{i=-\infty}^{\infty} \tan^{-1} \left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega'}{n^i} + \frac{n^i}{\omega'}} \right) \qquad (41)$$

and

$$\emptyset(k,n,n\omega') = \sum_{i=-\infty}^{\infty} \tan^{-1} \left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega'}{n^{i-1}} + \frac{n^{i-1}}{\omega'}} \right) \qquad (42)$$

In equation 42 let i-1=j. Equation 42 becomes

$$\emptyset(k,n,n\omega') = \sum_{j=-\infty}^{\infty} \tan^{-1} \left(\frac{k^{1/2} - k^{-1/2}}{\frac{\omega'}{n^j} + \frac{n^j}{\omega'}} \right)$$

A comparison of this result with equation 41 reveals that

$$\emptyset(k,n,\omega') = \emptyset(k,n,n\omega')$$

Thus, $\emptyset$ must be periodic as a function of log ω and must have a period of log n.

APPENDIX II

TABLES

Table 1. Evaluation of the Infinite-Band Phase Function (See Equation 5)

Value of θ in Degrees	Value of n	ϕ_{max} (ω=1) in Degrees	ϕ_{min} ($\omega=\sqrt{n}$) in Degrees	$\Delta\phi$ in Degrees	$\phi_m - \theta$ in Degrees
5	2	5.0001	5.0000	0.0001	0.0000
5	4	5.0161	4.9839	0.0322	0.0000
5	7	5.1255	4.8760	0.2495	0.0008
5	10	5.2774	4.7300	0.5474	0.0037
5	20	5.7649	4.2891	1.4758	0.0270
5	40	6.4636	3.7231	2.7405	0.0934
5	70	7.1389	3.2407	3.8982	0.1898
5	100	7.6064	2.9397	4.6667	0.2730
10	2	10.0001	10.0000	0.0001	0.0000
10	4	10.0317	9.9682	0.0635	0.0000
10	7	10.2472	9.7557	0.4915	0.0014
10	10	10.5461	9.4679	1.0782	0.0069
10	20	11.5038	8.5976	2.9062	0.0507
10	40	12.8727	7.4779	5.3948	0.1753
10	70	14.1906	6.5215	7.6691	0.3561
10	100	15.1000	5.9237	9.1763	0.5119
15	2	15.0001	15.0000	0.0001	0.0000
15	4	15.0463	14.9536	0.0927	0.0000
15	7	15.3612	14.6427	0.7185	0.0019
15	10	15.7975	14.2213	1.5762	0.0094
15	20	17.1921	12.9445	4.2476	0.0683
15	40	19.1759	11.2960	7.8799	0.2359
15	70	21.0745	9.8825	11.1920	0.4785
15	100	22.3777	8.9965	13.3812	0.6871
20	2	20.0001	19.9999	0.0002	0.0000
20	4	20.0596	19.9404	0.1192	0.0000
20	7	20.4640	19.5404	0.9236	0.0022
20	10	21.0238	18.9976	2.0262	0.0107
20	20	22.8071	17.3481	5.4590	0.0776
20	40	25.3277	15.2080	10.1197	0.2679
20	70	27.7212	13.3636	14.3576	0.5424
20	100	29.3527	12.2029	17.1498	0.7778
25	2	25.0001	25.0000	0.0001	0.0000
25	4	25.0710	24.9290	0.1420	0.0000
25	7	25.5526	24.4518	1.1008	0.0022
25	10	26.2180	23.8034	2.4146	0.0107
25	20	28.3294	21.8257	6.5037	0.0776
25	40	31.2908	19.2440	12.0468	0.2674
25	70	34.0767	17.0045	17.0722	0.5406
25	100	35.9602	15.5880	20.3722	0.7741

Table 1. (continued)

Value of θ in Degrees	Value of n	ϕ_{max} (ω=1) in Degrees	ϕ_{min} ($\omega=\sqrt{n}$) in Degrees	$\Delta\phi$ in Degrees	$\phi_m - \theta$ in Degrees
30	2	30.0001	30.0000	0.0001	0.0000
30	4	30.0803	29.9197	0.1606	0.0000
30	7	30.6241	29.3797	1.2444	0.0019
30	10	31.3742	28.6446	2.7296	0.0094
30	20	33.7434	26.3930	7.3504	0.0682
30	40	37.0373	23.4323	13.6050	0.2348
30	70	40.1040	20.8440	19.2600	0.4739
30	100	42.1585	19.1969	22.9616	0.6777
35	2	35.0001	35.0000	0.0001	0.0000
35	4	35.0871	34.9128	0.1743	0.0000
35	7	35.6766	34.3263	1.3503	0.0014
35	10	36.4878	33.5261	2.9617	0.0070
35	20	39.0374	31.0638	7.9736	0.0506
35	40	42.5490	27.7991	14.7499	0.1740
35	70	45.7822	24.9196	20.8626	0.3509
35	100	47.9282	23.0742	24.8540	0.5012
40	2	40.0001	39.9999	0.0002	0.0000
40	4	40.0913	39.9086	0.1827	0.0000
40	7	40.7083	39.2932	1.4151	0.0008
40	10	41.5556	38.4518	3.1038	0.0037
40	20	44.2045	35.8493	8.3552	0.0269
40	40	47.8173	32.3677	15.4496	0.0925
40	70	51.1064	29.2664	21.8400	0.1864
40	100	53.2690	27.2631	26.0059	0.2661
45	2	45.0001	45.0000	0.0001	0.0000
45	4	45.0927	44.9072	0.1855	0.0000
45	7	45.7184	44.2815	1.4369	0.0000
45	10	46.5758	43.4241	3.1517	0.0000
45	20	49.2418	40.7582	8.4836	0.0000
45	40	52.8425	37.1575	15.6850	0.0000
45	70	56.0842	33.9158	22.1684	0.0000
45	100	58.1963	31.8037	26.3926	0.0000

Table 2. Ratio of $\Delta\phi(\theta,n)$ to $\Delta\phi(45°,n)$ for Various Values of θ and n

Value of θ in Degrees	$\frac{2[\phi(45°,n,\omega_o) - 45°]}{\Delta\phi(45°,n)} = \frac{\Delta\phi(\theta,n)}{\Delta\phi(45°,n)}$					
	n = 4	n = 10	n = 20	n = 40	n = 70	n = 100
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.625	0.1949	0.1951	0.1954	0.1963	0.1975	0.1986
11.25	0.3825	0.3828	0.3833	0.3848	0.3869	0.3888
16.875	0.5555	0.5557	0.5563	0.5580	0.5605	0.5627
22.5	0.7070	0.7072	0.7078	0.7094	0.7117	0.7136
28.125	0.8314	0.8315	0.8319	0.8331	0.8348	0.8362
33.750	0.9238	0.9239	0.9241	0.9247	0.9256	0.9264
39.375	0.9807	0.9808	0.9808	0.9810	0.9813	0.9815
45	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3. Error in $\Delta\phi$ Caused by the Approximation of $\Delta\phi(\theta,n)/\Delta\phi(45°,n)$ with Sin 2θ

Value of θ in Degrees	$\Delta\phi$(actual) - $\Delta\phi$(calculated) in Degrees					
	n = 4	n = 10	n = 20	n = 40	n = 70	n = 100
0	0	0	0	0	0	0
5	0.0000	0.0001	0.0026	0.0168	0.0486	0.0836
10	0.0000	0.0002	0.0047	0.0302	0.0870	0.1496
15	0.0000	0.0003	0.0058	0.0374	0.1078	0.1849
20	0.0000	0.0003	0.0059	0.0376	0.1080	0.1850
25	0.0000	0.0002	0.0049	0.0315	0.0902	0.1543
30	0.0000	0.0002	0.0034	0.0215	0.0615	0.1050
35	0.0000	0.0001	0.0017	0.0109	0.0311	0.0531
40	0.0000	0.0000	0.0005	0.0029	0.0084	0.0143
45	0	0	0	0	0	0

Table 4. Error in $\Delta\phi$ Caused by the Approximation of $\Delta\phi(\theta,n)/\Delta\phi(45°,n)$ with Sin $2\phi_m$

Value of θ in Degrees	$\Delta\phi$(actual) - $\Delta\phi$(calculated) in Degrees					
	n = 4	n = 10	n = 20	n = 40	n = 70	n = 100
0	0	0	0	0	0	0
5	0.0000	-0.0003	-0.0052	-0.0335	-0.0959	-0.1639
10	0.0000	-0.0005	-0.0094	-0.0599	-0.1713	-0.2921
15	0.0000	-0.0006	-0.0116	-0.0742	-0.2114	-0.3595
20	0.0000	-0.0006	-0.0117	-0.0743	-0.2110	-0.3577
25	0.0000	-0.0005	-0.0098	-0.0621	-0.1756	-0.2967
30	0.0000	-0.0003	-0.0067	-0.0423	-0.1192	-0.2008
35	0.0000	-0.0002	-0.0034	-0.0214	-0.0602	-0.1010
40	0.0000	0.0000	-0.0009	-0.0058	-0.0162	-0.0271
45	0	0	0	0	0	0

Table 5. Variation of $\Delta\phi$ with n, for $\theta = 45°$

Value of n	$\Delta\phi(45°,n)$ in Degrees	Value of n	$\Delta\phi(45°,n)$ in Degrees	Value of n	$\Delta\phi(45°,n)$ in Degrees
3	0.0287	9.75	3.0051	26	11.0469
3.25	0.0529	10	3.1517	27	11.4343
3.5	0.0869	10.5	3.4447	28	11.8115
3.75	0.1310	11	3.7368	29	12.1790
4	0.1855	11.5	4.0272	30	12.5372
4.25	0.2499	12	4.3156	31	12.8865
4.5	0.3239	12.5	4.6014	32	13.2272
4.75	0.4067	13	4.8844	33	13.5598
5	0.4977	13.5	5.1643	34	13.8845
5.25	0.5960	14	5.4409	35	14.2017
5.5	0.7012	14.5	5.7141	36	14.5117
5.75	0.8123	15	5.9839	37	14.8148
6	0.9288	15.5	6.2500	38	15.1112
6.25	1.0501	16	6.5126	39	15.4011
6.5	1.1755	16.5	6.7715	40	15.6850
6.75	1.3046	17	7.0268	45	17.0193
7	1.4369	17.5	7.2784	50	18.2299
7.25	1.5720	18	7.5265	55	19.3361
7.5	1.7094	18.5	7.7710	60	20.3535
7.75	1.8488	19	8.0120	65	21.2942
8	1.9900	19.5	8.2495	70	22.1684
8.25	2.1326	20	8.4836	75	22.9843
8.5	2.2763	21	8.9417	80	23.7486
8.75	2.4210	22	9.3868	85	24.4670
9.	2.5664	23	9.8194	90	25.1446
9.25	2.7123	24	10.2400	95	25.7852
9.5	2.8586	25	10.6490	100	26.3925

Table 6. Difference between ϕ_m and θ, for $\theta = 22.5°$

Value of n	$\phi_m - 22.5°$ in Degrees	Value of n	$\phi_m - 22.5°$ in Degrees
5	0.0002	22	0.0965
5.5	0.0005	23	0.1057
6	0.0009	24	0.1150
6.5	0.0015	25	0.1244
7	0.0022	26	0.1339
7.5	0.0032	27	0.1436
8	0.0043	28	0.1533
8.5	0.0056	29	0.1630
9	0.0072	30	0.1728
9.5	0.0089	31	0.1827
10	0.0108	32	0.1926
10.5	0.0129	33	0.2025
11	0.0152	34	0.2124
11.5	0.0177	35	0.2223
12	0.0203	36	0.2322
12.5	0.0231	37	0.2421
13	0.0260	38	0.2520
13.5	0.0291	39	0.2619
14	0.0323	40	0.2718
14.5	0.0357	45	0.3207
15	0.0391	50	0.3688
15.5	0.0427	55	0.4158
16	0.0464	60	0.4616
16.5	0.0501	65	0.5063
17	0.0540	70	0.5499
17.5	0.0579	75	0.5922
18	0.0620	80	0.6335
18.5	0.0661	85	0.6736
19	0.0702	90	0.7127
19.5	0.0745	95	0.7508
20	0.0788	100	0.7879
21	0.0876		

Table 7. Error in ϕ_m Caused by the Use of Equation 23

Value of θ in Degrees	ϕ_m(actual) - ϕ_m(calculated) in Degrees			
	n = 20	n = 40	n = 70	n = 100
0	0	0	0	0
5	0.0000	0.0004	0.0017	0.0035
10	0.0000	0.0006	0.0026	0.0054
15	0.0000	0.0006	0.0023	0.0047
20	0.0000	0.0002	0.0009	0.0018
25	0.0000	-0.0002	-0.0009	-0.0018
30	0.0000	-0.0006	-0.0023	-0.0046
35	0.0000	-0.0006	-0.0026	-0.0053
40	0.0000	-0.0004	-0.0017	-0.0034
45	0	0	0	0

Table 8. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 5°$, $n \leq 8$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$					
	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2					-0.7512	-0.3013
-1				-0.7996	-0.1650	0.1647
0				-0.2180	0.2386	0.4722
1			-0.6234	0.1503	0.4673	0.6262
2			-0.1951	0.3337	0.5445	0.6474
3			0.0274	0.3729	0.5060	0.5687
4			0.1020	0.3153	0.3942	0.4297
5			0.0882	0.2096	0.2527	0.2710
6			0.0392	0.1001	0.1208	0.1291
7			-0.0005	0.0222	0.0297	0.0326
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.2312	0.1179	0.0273	0.0094	0.0041	
12	1.5646	0.1305	0.0271	0.0084	0.0034	
14	1.4751	0.1065	0.0198	0.0056	0.0021	
16	1.2255	0.0768	0.0128	0.0033	0.0012	
18	0.9490	0.0519	0.0078	0.0020	0.0006	
20	0.7030	0.0336	0.0046	0.0011	0.0003	
22	0.5057	0.0212	0.0026	0.0006	0.0002	
24	0.3562	0.0132	0.0015	0.0003	0.0001	
26	0.2531	0.0082	0.0009	0.0002	0.0000	
28	0.1750	0.0058	0.0007	0.0001		
30	0.1205	0.0037	0.0005	0.0001		
32	0.0826	0.0024	0.0003	0.0000		

Table 9. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 5°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$					
	n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7					-0.2788	-0.2045
-6			-0.6930	-0.2666	-0.1227	-0.0704
-5		-0.6415	-0.3488	-0.0300	0.0674	0.0984
-4	-0.8778	-0.2178	0.0023	0.2295	0.2885	0.3021
-3	-0.3217	0.1744	0.3353	0.4929	0.5265	0.5297
-2	0.1410	0.5038	0.6191	0.7293	0.7515	0.7528
-1	0.4845	0.7418	0.8223	0.9004	0.9195	0.9236
0	0.6950	0.8694	0.9225	0.9738	0.9878	0.9923
1	0.7742	0.8842	0.9147	0.9384	0.9398	0.9376
2	0.7396	0.8012	0.8138	0.8110	0.7960	0.7839
3	0.6216	0.6498	0.6499	0.6284	0.6020	0.5831
4	0.4570	0.4655	0.4595	0.4315	0.4033	0.3844
5	0.2836	0.2835	0.2768	0.2534	0.2318	0.2179
6	0.1343	0.1327	0.1286	0.1154	0.1039	0.0966
7	0.0345	0.0341	0.0329	0.0292	0.0260	0.0241
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 10. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 45°$, $n \leq 8$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$					
	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2					-0.8090	-0.3296
-1				-0.8624	-0.1908	0.1570
0				-0.2549	0.2257	0.4704
1				0.1263	0.4596	0.6260
2			-0.2427	0.3172	0.5400	0.6490
3			-0.0062	0.3619	0.5045	0.5726
4			0.0792	0.3085	0.3948	0.4350
5			0.0735	0.2058	0.2540	0.2759
6			0.0309	0.0979	0.1216	0.1320
7			-0.0041	0.0211	0.0297	0.0333
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.3438	0.1320	0.0312	0.0110	0.0048	
12	1.7272	0.1477	0.0313	0.0100	0.0041	
14	1.6434	0.1220	0.0232	0.0068	0.0026	
16	1.3761	0.0888	0.0151	0.0041	0.0014	
18	1.0724	0.0603	0.0093	0.0023	0.0008	
20	0.7983	0.0392	0.0055	0.0012	0.0004	
22	0.5760	0.0249	0.0031	0.0007	0.0002	
24	0.4066	0.0155	0.0018	0.0004	0.0001	
26	0.2835	0.0095	0.0010	0.0002	0.0000	
28	0.1948	0.0058	0.0006	0.0001		
30	0.1328	0.0035	0.0003	0.0001		
32	0.0900	0.0021	0.0002	0.0000		

Table 11. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 45°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$					
	n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7					-0.4443	-0.3445
-6			-0.8848	-0.3713	-0.1852	-0.1123
-5		-0.7605	-0.4241	-0.0394	0.0947	0.1457
-4	-0.9798	-0.2486	0.0018	0.2801	0.3733	0.4077
-3	-0.3529	0.1877	0.3677	0.5612	0.6231	0.6456
-2	0.1411	0.5280	0.6525	0.7809	0.8195	0.8330
-1	0.4925	0.7598	0.8424	0.9231	0.9452	0.9523
0	0.7018	0.8797	0.9321	0.9802	0.9919	0.9953
1	0.7799	0.8935	0.9253	0.9529	0.9593	0.9613
2	0.7472	0.8166	0.8350	0.8504	0.8544	0.8561
3	0.6323	0.6725	0.6825	0.6906	0.6929	0.6942
4	0.4692	0.4909	0.4959	0.4993	0.4998	0.4999
5	0.2938	0.3045	0.3067	0.3074	0.3065	0.3056
6	0.1402	0.1448	0.1455	0.1452	0.1442	0.1432
7	0.0361	0.0376	0.0378	0.0376	0.0372	0.0368
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 12. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 85°$, $n \leq 8$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$ n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2					-0.9376	-0.4006
-1				-1.0264	-0.2715	0.1159
0				-0.3716	0.1694	0.4421
1				0.0399	0.4178	0.6050
2			-0.3954	0.2532	0.5096	0.6346
3			-0.1172	0.3159	0.4838	0.5647
4			0.0021	0.2775	0.3822	0.4323
5			0.0237	0.1863	0.2469	0.2757
6			0.0025	0.0870	0.1177	0.1321
7			-0.0162	0.0164	0.0278	0.0329
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	1.7241	0.1781	0.0441	0.0161	0.0074	
12	2.2735	0.2070	0.0464	0.0156	0.0067	
14	2.2155	0.1776	0.0358	0.0112	0.0044	
16	1.8945	0.1330	0.0242	0.0069	0.0026	
18	1.5038	0.0923	0.0152	0.0040	0.0014	
20	1.1419	0.0612	0.0091	0.0022	0.0008	
22	0.8363	0.0393	0.0053	0.0012	0.0004	
24	0.5965	0.0247	0.0030	0.0006	0.0002	
26	0.4190	0.0153	0.0019	0.0003	0.0001	
28	0.2905	0.0094	0.0011	0.0002	0.0001	
30	0.2000	0.0065	0.0007	0.0002	0.0000	
32	0.1362	0.0041	0.0004	0.0001	0.0000	

Table 13. The Correction Function β_1 Relative to Its Tolerance Band ($\theta = 85°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_1(\omega)}{\Delta\phi}$					
	n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7					-0.9081	-0.7684
-6				-0.5718	-0.3139	-0.2028
-5		-0.9392	-0.5368	-0.0540	0.1361	0.2185
-4	-1.1341	-0.2945	-0.0024	0.3388	0.4696	0.5258
-3	-0.4113	0.1954	0.3989	0.6260	0.7083	0.7427
-2	0.1222	0.5464	0.6817	0.8226	0.8685	0.8864
-1	0.4850	0.7722	0.8582	0.9394	0.9614	0.9687
0	0.6961	0.8855	0.9387	0.9844	0.9944	0.9970
1	0.7757	0.8992	0.9332	0.9631	0.9715	0.9747
2	0.7472	0.8287	0.8528	0.8809	0.8950	0.9028
3	0.6374	0.6927	0.7123	0.7447	0.7677	0.7819
4	0.4776	0.5153	0.5319	0.5671	0.5963	0.6154
5	0.3021	0.3261	0.3386	0.3694	0.3974	0.4166
6	0.1452	0.1577	0.1649	0.1838	0.2021	0.2151
7	0.0374	0.0414	0.0436	0.0494	0.0552	0.0594
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 14. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 5°$, $n \leq 8$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$					
	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2					-0.9046	-0.3926
-1					-0.3041	0.0819
0				-0.4448	0.1160	0.3996
1			-1.0885	-0.0457	0.3624	0.5647
2			-0.5940	0.1684	0.4573	0.5969
3			-0.3086	0.2364	0.4354	0.5284
4			-0.1762	0.2051	0.3384	0.3985
5			-0.1386	0.1222	0.2094	0.2474
6			-0.1431	0.0320	0.0880	0.1116
7			-0.1452	-0.0301	0.0052	0.0199
8		-0.4793	-0.1137	-0.0397	-0.0180	-0.0091
10		-0.1951	-0.0409	-0.0128	-0.0054	
12	-0.9706	-0.0688	-0.0127	-0.0036	-0.0014	
14	-0.2983	-0.0181	-0.0030	-0.0007	-0.0003	
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
18	0.1101	0.0050	0.0006	0.0002	0.0000	
20	0.1331	0.0052	0.0006	0.0002	0.0000	
22	0.1208	0.0041	0.0005	0.0001	0.0000	
24	0.0973	0.0030	0.0003	0.0001	0.0000	
26	0.0798	0.0020	0.0002	0.0001	0.0000	
28	0.0592	0.0021	0.0004	0.0000	0.0001	
30	0.0432	0.0015	0.0003		0.0000	
32	0.0312	0.0011	0.0002			

Table 15. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 5°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$					
	n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7				-	-0.2907	-0.2048
-6				-0.2696	-0.1378	-0.0709
-5		-0.6538	-0.3588	-0.0336	0.0488	0.0979
-4	-0.9282	-0.2307	-0.0084	0.2255	0.2664	0.3014
-3	-0.3708	0.1615	0.3243	0.4887	0.5017	0.5290
-2	0.0947	0.4916	0.6085	0.7250	0.7258	0.7520
-1	0.4426	0.7307	0.8126	0.8965	0.8954	0.9229
0	0.6585	0.8599	0.9143	0.9706	0.9681	0.9917
1	0.7437	0.8764	0.9082	0.9359	0.9257	0.9372
2	0.7152	0.7953	0.8090	0.8093	0.7872	0.7836
3	0.6026	0.6454	0.6465	0.6273	0.5969	0.5830
4	0.4428	0.4625	0.4573	0.4309	0.4007	0.3843
5	0.2732	0.2814	0.2754	0.2530	0.2305	0.2178
6	0.1269	0.1314	0.1277	0.1152	0.1033	0.0966
7	0.0294	0.0332	0.0324	0.0291	0.0257	0.0241
8	-0.0035	-0.0005	-0.0003	0.0000	-0.0001	0.0000

Table 16. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 45°$, $n \leq 8$)

Value of i for the Frequency	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$					
$\omega = n^{i/16}$	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2					-0.9708	-0.4259
-1					-0.3338	0.0722
0				-0.4925	0.1017	0.3974
1				-0.0776	0.3541	0.5644
2			-0.6590	0.1451	0.4521	0.5982
3			-0.3578	0.2190	0.4326	0.5315
4			-0.2139	0.1918	0.3371	0.4026
5			-0.1677	0.1120	0.2086	0.2508
6			-0.1652	0.0236	0.0864	0.1129
7			-0.1616	-0.0369	0.0028	0.0191
8		-0.5243	-0.1252	-0.0447	-0.0201	-0.0104
10		-0.2159	-0.0456	-0.0147	-0.0061	
12	-1.0699	-0.0774	-0.0144	-0.0042	-0.0016	
14	-0.3313	-0.0205	-0.0034	-0.0009	-0.0003	
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
18	0.1237	0.0057	0.0008	0.0002	0.0000	
20	0.1500	0.0059	0.0007	0.0001		
22	0.1361	0.0048	0.0005	0.0001		
24	0.1097	0.0034	0.0003	0.0001		
26	0.0840	0.0022	0.0002	0.0000		
28	0.0612	0.0014	0.0001			
30	0.0436	0.0009	0.0001			
32	0.0304	0.0006	0.0000			

Table 17. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 45°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$ n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7					-0.4438	-0.3444
-6				-0.3731	-0.1846	-0.1122
-5		-0.7753	-0.4306	-0.0412	0.0954	0.1458
-4	-1.0328	-0.2627	-0.0045	0.2783	0.3739	0.4078
-3	-0.4017	0.1746	0.3619	0.5595	0.6237	0.6456
-2	0.0972	0.5163	0.6473	0.7794	0.8200	0.8331
-1	0.4541	0.7497	0.8380	0.9218	0.9456	0.9524
0	0.6691	0.8712	0.9284	0.9792	0.9923	0.9954
1	0.7527	0.8866	0.9224	0.9521	0.9596	0.9613
2	0.7251	0.8112	0.8328	0.8498	0.8546	0.8561
3	0.6148	0.6684	0.6809	0.6902	0.6930	0.6942
4	0.4557	0.4879	0.4947	0.4990	0.4999	0.4999
5	0.2837	0.3024	0.3059	0.3072	0.3065	0.3056
6	0.1328	0.1433	0.1450	0.1452	0.1442	0.1432
7	0.0308	0.0366	0.0374	0.0376	0.0372	0.0368
8	-0.0037	-0.0006	-0.0002	0.0000	0.0000	0.0000

Table 18. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 85°$, $n \leq 8$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$					
	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
-2						-0.5098
-1					-0.4289	0.0221
0				-0.6400	0.0341	0.3620
1				-0.1911	0.3023	0.5370
2			-0.8775	0.0558	0.4118	0.5776
3			-0.5306	0.1488	0.4018	0.5173
4			-0.3494	0.1374	0.3142	0.3934
5			-0.2725	0.0701	0.1913	0.2442
6			-0.2447	-0.0082	0.0728	0.1069
7			-0.2206	-0.0608	-0.0080	0.0132
8		-0.6804	-0.1672	-0.0617	-0.0281	-0.0152
10		-0.2927	-0.0645	-0.0220	-0.0092	
12	-1.4041	-0.1096	-0.0215	-0.0068	-0.0026	
14	-0.4444	-0.0298	-0.0052	-0.0015	-0.0005	
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
18	0.1736	0.0087	0.0012	0.0003	0.0001	
20	0.2189	0.0094	0.0012	0.0003	0.0001	
22	0.2014	0.0075	0.0008	0.0002	0.0001	
24	0.1633	0.0054	0.0006	0.0001	0.0001	
26	0.1254	0.0036	0.0006	0.0001	0.0000	
28	0.0927	0.0024	0.0004	0.0000		
30	0.0670	0.0023	0.0003	0.0001		
32	0.0470	0.0016	0.0002	0.0001		

Table 19. The Correction Function β_2 Relative to Its Tolerance Band ($\theta = 85°$, $n \geq 10$)

Value of i for the Frequency $\omega = n^{i/16}$	$\frac{\alpha''(\omega) - \beta_2(\omega)}{\Delta\phi}$					
	n = 10	n = 15	n = 20	n = 40	n = 70	n = 100
-7					-0.9094	-0.7788
-6				-0.5776	-0.3149	-0.2108
-5		-0.9589	-0.5488	-0.0586	0.1354	0.2125
-4	-1.1998	-0.3111	-0.0123	0.3351	0.4690	0.5213
-3	-0.4676	0.1815	0.3906	0.6231	0.7079	0.7393
-2	0.0741	0.5348	0.6749	0.8202	0.8681	0.8838
-1	0.4442	0.7625	0.8526	0.9376	0.9611	0.9668
0	0.6616	0.8774	0.9342	0.9830	0.9942	0.9955
1	0.7469	0.8927	0.9295	0.9620	0.9713	0.9736
2	0.7232	0.8234	0.8499	0.8800	0.8949	0.9020
3	0.6178	0.6884	0.7100	0.7441	0.7676	0.7813
4	0.4618	0.5120	0.5301	0.5666	0.5962	0.6150
5	0.2894	0.3235	0.3373	0.3690	0.3973	0.4163
6	0.1353	0.1558	0.1639	0.1835	0.2021	0.2150
7	0.0298	0.0399	0.0428	0.0492	0.0552	0.0593
8	-0.0057	-0.0010	-0.0005	-0.0001	0.0000	-0.0001

Table 20. Normalized Tolerance Band for the Correction Functions

Value of θ in Degrees	Value of n	$\frac{\phi_{max} - \phi(\omega)}{\Delta\phi}$				
		$\omega = n^{\frac{24}{32}}$	$\omega = n^{\frac{28}{32}}$	$\omega = n^{\frac{29}{32}}$	$\omega = n^{\frac{30}{32}}$	$\omega = n^{\frac{31}{32}}$
5	4	0.5012	0.1470	0.0846	0.0382	0.0096
5	7	0.5062	0.1496	0.0862	0.0390	0.0098
5	10	0.5135	0.1534	0.0885	0.0401	0.0101
5	20	0.5365	0.1657	0.0962	0.0438	0.0111
5	40	0.5678	0.1839	0.1078	0.0494	0.0126
5	70	0.5965	0.2021	0.1196	0.0552	0.0141
5	100	0.6155	0.2152	0.1281	0.0594	0.0152
25	4	0.5004	0.1466	0.0845	0.0382	0.0096
25	7	0.5040	0.1485	0.0855	0.0386	0.0098
25	10	0.5088	0.1509	0.0870	0.0394	0.0099
25	20	0.5239	0.1586	0.0918	0.0416	0.0105
25	40	0.5443	0.1694	0.0985	0.0448	0.0114
25	70	0.5634	0.1795	0.1048	0.0478	0.0122
25	100	0.5762	0.1865	0.1091	0.0499	0.0127
45	4	0.4999	0.1464	0.0843	0.0381	0.0096
45	7	0.5000	0.1464	0.0843	0.0381	0.0096
45	10	0.5000	0.1464	0.0842	0.0380	0.0096
45	20	0.5000	0.1461	0.0840	0.0379	0.0096
45	40	0.5000	0.1453	0.0834	0.0376	0.0095
45	70	0.5000	0.1441	0.0826	0.0372	0.0094
45	100	0.5000	0.1432	0.0819	0.0368	0.0093
65	4	0.4996	0.1462	0.0841	0.0380	0.0096
65	7	0.4960	0.1444	0.0830	0.0375	0.0095
65	10	0.4912	0.1420	0.0816	0.0368	0.0093
65	20	0.4761	0.1347	0.0770	0.0346	0.0087
65	40	0.4556	0.1250	0.0711	0.0318	0.0080
65	70	0.4366	0.1163	0.0658	0.0294	0.0074
65	100	0.4238	0.1106	0.0624	0.0278	0.0070
85	4	0.4989	0.1462	0.0842	0.0381	0.0097
85	7	0.4938	0.1434	0.0824	0.0372	0.0094
85	10	0.4865	0.1398	0.0802	0.0361	0.0091
85	20	0.4635	0.1291	0.0736	0.0330	0.0083
85	40	0.4322	0.1155	0.0654	0.0292	0.0073
85	70	0.4035	0.1039	0.0585	0.0260	0.0065
85	100	0.3845	0.0966	0.0542	0.0241	0.0060

Table 21. Variation of Bandwidth with Phase Tolerance and Network Complexity, for $\theta = 45°$ $(r \leq 6)$

$\frac{\Delta\phi(45°)}{2}$ in Degrees	Value of n	Value of a	$B = \frac{\text{Upper Frequency Limit}}{\text{Lower Frequency Limit}} = n^{r-1-2a}$				
			r = 2	r = 3	r = 4	r = 5	r = 6
For the Correction Function β_1							
0.0928	4	0.500	1.19	4.00	16.0	64.0	256
0.2488	5	0.250	2.24	11.20	56.0	280	1,400
0.4644	6	0.0530	4.93	29.7	178.0	1,060	6,400
0.7184	7	-0.0275	7.80	54.2	380	2,700	19,000
0.9950	8	-0.0795	11.15	89.0	708	5,660	45,300
1.5759	10	-0.1438	19.40	195	1,950	19,500	
4.2418	20	-0.250	90.0	1,800	36,000		
7.8425	40	-0.303	373	15,000			
1.0842	70	-0.331	1,170	81,900			
13.1963	100	-0.3436	2,350				
For the Correction Function β_2							
0.0144	3	0.813		1.51	4.53	13.6	40.8
0.0928	4	0.675		2.46	9.83	39.3	157
0.2488	5	0.600		3.62	18.1	90.5	450

Table 22. Variation of Bandwidth with Phase Tolerance and Network Complexity, for $\theta = 45°$ $(r \geq 7)$

$\frac{\Delta\phi(45°)}{2}$ in Degrees	Value of n	Value of a	$B = \frac{\text{Upper Frequency Limit}}{\text{Lower Frequency Limit}} = n^{r-1-2a}$				
			r = 7	r = 8	r = 9	r = 10	r = 11
For the Correction Function β_1							
0.0928	4	0.500	1,010	4,100	16,200	64,800	
0.2488	5	0.2500	7,000	35,000			
0.4644	6	0.0530	38,200				
For the Correction Function β_2							
0.0144	3	0.813	122	365	1,100	3,300	10,000
0.0928	4	0.675	630	2,500	10,000	40,000	
0.2488	5	0.600	2,250	11,200	56,000		

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